Properties of several fuzzy set spaces ☆

Huan Huang

Department of Mathematics, Jimei University, Xiamen 361021, China

Abstract

This paper discusses the properties the spaces of fuzzy sets in a metric space equipped with the endograph metric and the sendograph metric, respectively. We fist give some relations among the endograph metric, the sendograph metric and the Γ-convergence, and then investigate the level characterizations of the endograph metric and the Γ-convergence. By using the above results, we give some relations among the endograph metric, the sendograph metric, the supremum metric and the d_p^* metric, $p \geq 1$. On the basis of the above results, we present the characterizations of total boundedness, relative compactness and compactness in the space of compact positive α-cuts fuzzy sets equipped with the endograph metric, and in the space of compact support fuzzy sets equipped with the sendograph metric, respectively. Furthermore, we give completions of these metric spaces, respectively.

Keywords: Endograph metric; Sendograph metric; Hausdorff metric; Total boundedness; Relative compactness; Compactness; Completion

1. Introduction

A fuzzy set can be identified with its endograph. Also, a fuzzy set can be identified with its sendograph. So convergence structures on fuzzy sets can be defined on their endographs or sendographs. The the endograph metric $H_{\rm end}$ convergence, the sendograh metric $H_{\rm send}$ convergence and the Γ-convergence are this kind of convergence structures. These three convergence structures are related to each other [10, 18].

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Email address: `hhuangjy@126.com` (Huan Huang)

The endograph metric on fuzzy sets is shown to has significant advantages [16, 17]. The sendograph metric has attracted deserving attentions [3, 6]. Compactness is one of the central concepts in topology and analysis and useful in applications (see [14, 22]). There is a lot of work devoted to characterizations of compactness in various fuzzy set spaces endowed with different topologies [3, 6, 7, 9, 10, 19, 20, 24]. It is natural to consider what the completion of a metric space is. The recent results on completions of fuzzy set spaces including [9, 10].

In [10], we presented the relations and level characterizations of the endograph metric $H_{\rm end}$ and the Γ-convergence. Based on this, we have given the characterizations of total boundedness, relative compactness and compactness of fuzzy set spaces equipped with the endograph metric $H_{\rm end}$. We also pointed out the completions of fuzzy set spaces according to the endograph metric $H_{\rm end}$.

The common fuzzy sets used in theoretical research and practical applications are fuzzy sets in a metric space whose positive cut set are nonempty compact sets. Common compact fuzzy sets are common fuzzy sets whose support sets are compact. Throughout this paper, we suppose that X is a metric space endowed with a metric d. The symbols $F^1_{USCG}(X)$ and $F^1_{USCB}(X)$ are used to denote the set of common fuzzy sets in X and the set of common compact fuzzy sets in X, respectively. We use $F^1_{USC}(X)$ to denote the set of normal and upper semi-continuous fuzzy sets in X. $F^1_{USCB}(X)$ is a subset of $F^1_{USCG}(X)$. $F^1_{USCG}(X)$ is a subset of $F^1_{USC}(X)$.

The results in [10] are obtained on the realm of fuzzy sets in the m-dimensional Euclidean space $\mathbb{R}^m$. $\mathbb{R}^m$ is a special type of metric space (for simplicity, in this paper, $\mathbb{R}$ is used to denote the 1-dimensional Euclidean space). Of course, it is worth to study the fuzzy sets in a metric space [6, 7, 13]. In this paper, the results are obtained on the realm of fuzzy sets in a general metric space X. We mainly discuss $H_{\rm end}$ metirc and $H_{\rm send}$ metric on $F^1_{USC}(X)$ including the relations among $H_{\rm end}$ metric, $H_{\rm send}$ metric and other convergence structures, and properties of $H_{\rm end}$ metric and $H_{\rm send}$ metric.

The first part of this paper is the relations among the $H_{\rm end}$ metric, the $H_{\rm send}$ metric and the Γ-convergence, the level characterizations of the endograph metric $H_{\rm end}$ and the Γ-convergence, and the relations among the supremum metric d_∞, the $H_{\rm end}$ metric, the $H_{\rm send}$ metric and the d_p^* metric. The d_p^* metric is an expansion of the L_p-type d_p distance on $F^1_{USC}(X)$. The conclusions on the relations among the $H_{\rm end}$ metric, the $H_{\rm send}$ metric and the d_p^* metric are verified using the above results including the level characteri-

zation of the endograph metric H_{end}.

To aid discussion, we introduce the sets $P^1_{USC}(X)$ and $P^1_{USCB}(X)$. $P^1_{USCB}(X)$ is a subset of $P^1_{USC}(X)$. The $F^1_{USC}(X)$ and $F^1_{USCB}(X)$ can be viewed as the subsets of $P^1_{USC}(X)$ and $P^1_{USCB}(X)$, respectively.

We define the H_{send} distance and the H_{end} distance on $P^1_{USC}(X)$, and give the relations among the H_{send} distance, the H_{end} distance and the Kuratowski convergence on $P^1_{USC}(X)$. Then, as corollaries, we obtain the relations among the H_{send} metric, the H_{end} metric and the Γ-convergence on $F^1_{USC}(X)$.

We discuss the level characterizations of the Γ-convergence and the endograph metric H_{end} on fuzzy sets in $F^1_{USC}(X)$. It is shown that under some conditions, the Γ-convergence of fuzzy sets can be decomposed to the Kuratowski convergence of certain α-cuts, and the H_{end} metric convergence of fuzzy sets can be decomposed to the Hausdorff metric convergence of certain α-cuts.

The understanding of the relation among the H_{end} metric, the H_{send} metric and the Γ-convergence is beneficial for the understanding of themselves. The level characterizations help to study these three convergence structures on fuzzy sets by using the properties of the corresponding α-cuts.

A H_{send} metric convergent sequence is a H_{end} metric convergent sequence. A H_{end} metric convergent sequence is also a Γ-convergent sequence. So the knowledge of the Γ-convergent sequences can help us to analyse the properties of the H_{end} convergent sequences and the H_{send} convergent sequences. For this reason, we give the level characterizations of a Γ-convergent sequence in this paper.

Based on the results in the first part, we give the other results of this paper. The second part of this paper is the characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X), H_{\mathrm{end}})$ and $(F^1_{USCB}(X), H_{\mathrm{end}})$, respectively. Here we mention that the characterization of relatively compactness in $(F^1_{USCB}(X), H_{\mathrm{send}})$ has already been given by Greco [6].

The total boundedness is the key property of compactness in metric space. We show that a set U in $(F^1_{USCG}(X), H_{\mathrm{end}})$ is totally bounded (respectively, relatively compact) if and only if for each $\alpha \in (0, 1]$, the union of all the α-cuts of $u \in U$ is totally bounded (respectively, relatively compact) in (X, d). We also show that a set U in $(F^1_{USCB}(X), H_{\mathrm{send}})$ is totally bounded if and only if the union of all the 0-cuts of $u \in U$ is totally bounded in (X, d).

It is shown that for a set U in $(F^1_{USCG}(X), H_{\mathrm{end}})$ or $(F^1_{USCB}(X), H_{\mathrm{send}})$, the total boundedness, relative compactness and compactness of U are closely related to the total boundedness, relative compactness and compactness of

the union of all the α-cuts of $u \in U$ in (X, d), respectively.

We point out that some part of the proof of the characterizations in this paper is similar to the corresponding part in [10]. But in general, since a set in X need not has the properties of the set in $\mathbb{R}^m$, the proof of the conclusions in this paper requires deep understandings of the problem.

The third part is the completions of several common fuzzy set spaces under the H_{end} metric and the H_{send} metric, respectively.

Let $\widetilde{X}$ denote the completion of X. We show that the space $(P^1_{USCB}(\widetilde{X}), H_{\text{send}})$ is a completion of the fuzzy set space $(F^1_{USCB}(X), H_{\text{send}})$. Then we show that $(F^1_{USCG}(\widetilde{X}), H_{\text{end}})$ is a completion of $(F^1_{USCB}(X), H_{\text{end}})$. So, of course, $(F^1_{USCG}(\widetilde{X}), H_{\text{end}})$ is also a completion of $(F^1_{USCG}(X), H_{\text{end}})$.

These conclusions indicate that in the case of the H_{end} metric, a completion of the space of common compact fuzzy set in X is the space of common fuzzy set in $\widetilde{X}$; in the case of the H_{send} metric, a completion of the space of common compact fuzzy set in X is a metric space in which the space of common compact fuzzy set in $\widetilde{X}$ can be isometrically embedded, and each element of which is a nonempty compact set in $\widetilde{X} \times [0, 1]$.

The conclusions for the completions of the spaces of fuzzy set in X given in this paper apply to not only the cases that X is a complete metric space but also the cases that X is an incomplete metric space.

The remainder of this paper is organized as follows. In Section 2, we recall and give some basic notions and fundamental results related to fuzzy sets and convergence structures on them. In Section 3, we discuss the properties and relations of H_{send}, H_{end} and Kuratowski convergence on $P^1_{USC}(X)$. Based on this, we give some relations among H_{end}, H_{send} and Γ-convergence on $F^1_{USC}(X)$. In Sections 4 and 5, we investigate the level characterizations of the Γ-convergence and the H_{end} convergence, respectively. By using the above results, Section 6 discusses some relations among the d_∞ metric, the d_p^* metric, the H_{end} metric and the H_{send} metric. In Section 7, on the basis of the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X), H_{\text{end}})$ and $(F^1_{USCB}(X), H_{\text{send}})$, respectively. In Section 8, we give completions of $(F^1_{USCG}(X), H_{\text{end}})$ and $(F^1_{USCB}(X), H_{\text{send}})$, respectively. At last, we draw the conclusions in Section 9.

2. Fuzzy sets and convergence structures on them

In this section, we recall and give some basic notions and fundamental results related to fuzzy sets and convergence structures on them. Readers can refer to [2, 15, 23] for related contents.

A fuzzy set u in X can be seen as a function $u : X \to [0, 1]$. A subset S of X can be seen as a fuzzy set in X. If there is no confusion, the fuzzy set corresponding to S is often denoted by χ_S; that is,

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For simplicity, for $x \in X$, we will use $\widehat{x}$ to denote the fuzzy set $\chi_{\{x\}}$ in X. In this paper, if we want to emphasize a specific metric space X, we will write the fuzzy set corresponding to S in X as $S_{F(X)}$, and the fuzzy set corresponding to $\{x\}$ in X as $\widehat{x}_{F(X)}$.

The symbol $F(X)$ is used to denote the set of all fuzzy sets in X. For $u \in F(X)$ and $\alpha \in [0, 1]$, let $\{u > \alpha\}$ denote the set $\{x \in X : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α-*cut* of u, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the topological closure of S in (X, d).

For $u \in F(X)$, define

$$\operatorname{end} u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\},$$
$$\operatorname{send} u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\} \cap ([u]_0 \times [0, 1]).$$

$\operatorname{end} u$ and $\operatorname{send} u$ are called the endograph and the sendograph of u, respectively.

The symbol $K(X)$ and $C(X)$ are used to denote the set of all nonempty compact subsets of X and the set of all nonempty closed subsets of X, respectively.

Let $F^1_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e.,

$$F^1_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}.$$

We introduce some subclasses of $F^1_{USC}(X)$, which will be discussed in this paper. Define

$$F^1_{USCB}(X) := \{u \in F^1_{USC}(X) : [u]_0 \in K(X)\},$$
$$F^1_{USCG}(X) := \{u \in F^1_{USC}(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0,1]\}.$$

Clearly,

$$F^1_{USCB}(X) \subseteq F^1_{USCG}(X) \subseteq F^1_{USC}(X).$$

The set of (compact) fuzzy numbers are denoted by E^m. It is defined as

$$E^m := \{u \in F^1_{USCB}(\mathbb{R}^m) : [u]_\alpha \text{ is a convex subset of } \mathbb{R}^m \text{ for } \alpha \in [0,1]\}.$$

Fuzzy numbers have attracted much attention from theoretical research and practical applications [1, 2, 5, 8, 21, 23].

Let (X, d) be a metric space. We use $\boldsymbol{H}$ to denote the ***Hausdorff distance*** on $C(X)$ induced by d, i.e.,

$$\boldsymbol{H(U,V)} = \max\{H^*(U,V),\ H^*(V,U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U,V) = \sup_{u\in U} d\,(u,V) = \sup_{u\in U} \inf_{v\in V} d\,(u,v).$$

The metric $\overline{d}$ on $X \times [0,1]$ is defined as

$$\overline{d}((x,\alpha),(y,\beta)) = d(x,y) + |\alpha - \beta|.$$

If there is no confusion, we also use H to denote the Hausdorff distance on $C(X \times [0,1])$ induced by $\overline{d}$.

Remark 2.1. ρ is said to be a *metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be a metric space.

ρ is said to be an *extended metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be an extended metric space.

We can see that for arbitrary metric space (X, d), the Hausdorff distance H on $K(X)$ induced by d is a metric. So the Hausdorff distance H on $K(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$ is a metric. In these cases, we call the Hausdorff distance the Hausdorff metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric, but probably not a metric, because $H(A, B)$ could be equal to $+\infty$ for certain metric space X and $A, B \in C(X)$. Clearly, if H on $C(X)$ induced by d is not a metric, then H on $C(X \times [0, 1])$ induced by $\overline{d}$ is also not a metric. So the Hausdorff distance H on $C(X \times [0, 1])$ induced by $\overline{d}$ on $X \times [0, 1]$ is an extended metric but probably not a metric. In the cases that the Hausdorff distance H is an extended metric, we call the Hausdorff distance the Hausdorff extended metric.

We can see that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is H on $C(\mathbb{R}^m \times [0, 1])$.

In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric.

The Hausdorff metric has the following important properties.

Theorem 2.2. *[15, 19] Let (X, d) be a metric space and let H be the Hausdorff metric induced by d. Then*
(i) (X, d) is complete $\Longleftrightarrow$ $(K(X), H)$ is complete;
(ii) (X, d) is separable $\Longleftrightarrow$ $(K(X), H)$ is separable;
(iii) (X, d) is compact $\Longleftrightarrow$ $(K(X), H)$ is compact.

Rojas-Medar and Román-Flores [18] introduced the Kuratowski convergence of a sequence of sets in a metric space.

Let (X, d) be a metric space. Let C be a set in X and $\{C_n\}$ a sequence of sets in X. $\{C_n\}$ is said to ***Kuratowski converge*** to C according to (X, d), if

$$C = \liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n,$$

where

$$\liminf_{n\to\infty} C_n = \{x \in X : \ x = \lim_{n\to\infty} x_n, x_n \in C_n\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X : \ x = \lim_{j\to\infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} C_m}.$$

In this case, we'll write $\boldsymbol{C = \lim^{(K)}_{n\to\infty} C_n}$ according to (X, d). If there is no confusion, we will not emphasize the metric space (X, d) and write $\{C_n\}$ ***Kuratowski converges*** to C or $\boldsymbol{C = \lim^{(K)}_{n\to\infty} C_n}$ for simplicity.

Remark 2.3. Definition 3.1.4 in [15] gives the definitions of $\liminf C_n$, $\limsup C_n$ and $\lim C_n$ for a net of subsets $\{C_n, n \in D\}$ in a topological space. When $\{C_n, n = 1, 2, \ldots\}$ is a sequence of subsets of a metric space, $\liminf C_n$, $\limsup C_n$ and $\lim C_n$ according to Definition 3.1.4 in [15] are $\liminf_{n\to\infty} C_n$, $\limsup_{n\to\infty} C_n$ and $\lim^{(K)}_{n\to\infty} C_n$ according to the above definitions, respectively.

Rojas-Medar and Román-Flores [18] have introduced the Γ-convergence on $F^1_{USCB}(\mathbb{R}^m)$ by using the Kuratowski convergence. Similarly, we can define the Γ-convergence of a sequence of fuzzy sets on $F^1_{USC}(X)$.

Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. $\{u_n\}$ is said to **Γ-*converge*** to u, denoted by $\boldsymbol{u = \lim^{(\Gamma)}_{n\to\infty} u_n}$, if $\mathrm{end}\, u = \lim^{(K)}_{n\to\infty} \mathrm{end}\, u_n$ according to $(X \times [0,1], \overline{d})$.

Let (X, d) be a metric space and let $u \in F_(X)$. Then from basic analysis, the following three properties are equivalent:

(i) u is upper semi-continuous;

(ii) $\mathrm{end}\, u$ is closed in $(X \times [0,1], \overline{d})$;

(iii) $\mathrm{send}\, u$ is closed in $(X \times [0,1], \overline{d})$.

The endograph metric H_{end} and the sendograph metric H_{send} can be defined on $F^1_{USC}(X)$ as usual. For $u, v \in F^1_{USC}(X)$,

$$\boldsymbol{H_{\mathrm{end}}(u,v)} := H(\mathrm{end}\, u, \mathrm{end}\, v),$$
$$\boldsymbol{H_{\mathrm{send}}(u,v)} := H(\mathrm{send}\, u, \mathrm{send}\, v).$$

The endograph metric H_{end} and the sendograph metric H_{send} are defined by using the Hausdorff metric on $C(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$.

The d_∞ metric on $F^1_{USC}(X)$ is defined as

$$d_\infty(u,v) := \sup\{H([u]_\alpha, [v]_\alpha) : \alpha \in [0,1]\}.$$

Proposition 2.4. *Let $u, v \in F^1_{USC}(X)$. Then*

$$d_\infty(u,v) \geq H_{\mathrm{send}}(u,v) \geq H_{\mathrm{end}}(u,v). \tag{1}$$

Proof. (The proof is routine.) Note that for each $(x, \alpha) \in \mathrm{send}\, u$,

$$\overline{d}((x,\alpha), \mathrm{send}\, v) = \inf_{(y,\beta)\in \mathrm{send}\, v} \overline{d}((x,\alpha),(y,\beta))$$

$$
\begin{aligned}
&\le \inf_{(y,\alpha)\in \operatorname{send} v} \overline{d}((x,\alpha),(y,\alpha)) \\
&= \inf_{(y,\alpha)\in \operatorname{send} v} d(x,y) \\
&= \inf_{y\in [v]_\alpha} d(x,y) = d(x,[v]_\alpha).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&H^*(\operatorname{send} u, \operatorname{send} v) = \sup_{(x,\alpha)\in \operatorname{send} u} \overline{d}((x,\alpha), \operatorname{send} v) = \sup_{\alpha\in[0,1]} \sup_{(x,\alpha)\in \operatorname{send} u} \overline{d}((x,\alpha), \operatorname{send} v) \\
&\le \sup_{\alpha\in[0,1]} \sup_{(x,\alpha)\in \operatorname{send} u} d(x,[v]_\alpha) = \sup_{\alpha\in[0,1]} \sup_{x\in[u]_\alpha} d(x,[v]_\alpha) \\
&\le \sup_{\alpha\in[0,1]} H^*([u]_\alpha,[v]_\alpha) \le d_\infty(u,v). \qquad (2)
\end{aligned}
$$

Similarly $H^*(\operatorname{send} v, \operatorname{send} u) \le d_\infty(v,u) = d_\infty(u,v)$ (Clearly $H^*(\operatorname{send} v, \operatorname{send} u) \le d_\infty(v,u)$ can also be obtained by interchanging u with v in $H^*(\operatorname{send} u, \operatorname{send} v) \le d_\infty(u,v)$ in (2)). So we have

$$
H_{\text{send}}(u,v) = \max\{H^*(\operatorname{send} u, \operatorname{send} v), H^*(\operatorname{send} v, \operatorname{send} u)\} \le d_\infty(u,v).
$$

The conclusion $H_{\text{end}}(u,v) \le H_{\text{send}}(u,v)$ follows from Theorem 3.1 (i). □

Remark 2.5. We can see that H_{end} is a metric on $F^1_{USC}(X)$ with $H_{\text{end}}(u,v) \le 1$ for all $u,v \in F^1_{USC}(X)$. Both d_∞ and H_{send} are metrics on $F^1_{USCB}(X)$. However, each one of d_∞ and H_{send} on $F^1_{USC}(X)$ is an extended metric but probably not a metric. See also Remark 3.3 in [11].

We can see that both d_∞ and H_{send} on $F^1_{USCG}(\mathbb{R}^m)$ are not metrics, they are extended metrics.

For simplicity, in this paper, we call H_{send} on $F^1_{USC}(X)$ the H_{send} metric or the sendograph metric H_{send}. We call d_∞ on $F^1_{USC}(X)$ the d_∞ metric or the supremum metric d_∞.

3. H_{end}, H_{send} and Kuratowski convergence on $P^1_{USC}(X)$

In this section, we introduce $P^1_{USC}(X)$ and its subset $P^1_{USCB}(X)$, and define the H_{send} distance and the H_{end} distance on $P^1_{USC}(X)$. We discuss the properties and relations of H_{send}, H_{end} and Kuratowski convergence on $P^1_{USC}(X)$. Based on this, we give some relations among H_{end}, H_{send} and Γ-convergence on $F^1_{USC}(X)$.

Since $F^1_{USC}(X)$ and $F^1_{USCB}(X)$ can be seen as subsets of $P^1_{USC}(X)$ and $P^1_{USCB}(X)$, respectively, the conclusions on $P^1_{USC}(X)$ and $P^1_{USCB}(X)$ are useful for the discussions of fuzzy sets in this paper.

For $u \subseteq X \times [0,1]$ and $\alpha \in [0,1]$, let $\langle u\rangle_\alpha := \{x : (x,\alpha) \in u\}$. $P^1_{USC}(X)$ and $P^1_{USCB}(X)$ are subsets of the power set of $X \times [0,1]$ defined by

$$\begin{aligned} P^1_{USC}(X) := \{u \subseteq X \times [0,1] : \langle u\rangle_\alpha = \bigcap_{\beta<\alpha} \langle u\rangle_\beta \text{ for all } \alpha \in (0,1];\\ \langle u\rangle_\alpha \in C(X) \text{ for all } \alpha \in [0,1]\},\\ P^1_{USCB}(X) := \{u \in P^1_{USC}(X) : \langle u\rangle_\alpha \in K(X) \text{ for all } \alpha \in [0,1]\}. \end{aligned}$$

Clearly $P^1_{USCB}(X) \subseteq P^1_{USC}(X)$.

From the basic analysis, we can obtain that
(i) if $u \in P^1_{USC}(X)$, then $u \in C(X \times [0,1])$;
(ii) if $u \in C(X \times [0,1])$, then $\langle u\rangle_\alpha \in C(X) \cup \{\emptyset\}$ for all $\alpha \in [0,1]$;
(iii) if $u \in P^1_{USCB}(X)$, then $u \in K(X \times [0,1])$;
(iv) if $u \in K(X \times [0,1])$, then $\langle u\rangle_\alpha \in K(X) \cup \{\emptyset\}$ for all $\alpha \in [0,1]$.

So we have

$$P^1_{USC}(X) = \{u \in C(X \times [0,1]) : \langle u\rangle_1 \neq \emptyset, \langle u\rangle_\alpha = \bigcap_{\beta<\alpha} \langle u\rangle_\beta \text{ for all } \alpha \in (0,1]\},$$
$$P^1_{USCB}(X) = \{u \in P^1_{USC}(X) : u \in K(X \times [0,1])\}.$$

We define the $H_{\rm send}$ distance and the $H_{\rm end}$ distance on $P^1_{USC}(X)$:

$$\begin{aligned} H_{\rm send}(u,v) &:= H(u,v),\\ H_{\rm end}(u,v) &:= H(\underline{u},\underline{v}), \end{aligned}$$

where H is the Hausdorff metric on $C(X \times [0,1])$ induced by $\overline{d}$ on $X \times [0,1]$, and $\underline{u} := u \cup (X \times \{0\})$. Clearly, $H_{\rm send}$ is an extended metric on $P^1_{USC}(X)$ and $H_{\rm end} \leq 1$ on $P^1_{USC}(X)$.

If X contains more than one point, then $H_{\rm end}$ is a pseudometric on $P^1_{USC}(X)$; that is, $H_{\rm end}$ satisfies symmetry and triangle inequality, and for all $u, v \in P^1_{USC}(X)$, $0 \leq H_{\rm end}(u,v) < +\infty$ and $H_{\rm end}(u,u) = 0$. However, at this time, $H_{\rm end}$ is a not a metric on $P^1_{USC}(X)$ because $H_{\rm end}$ do not satisfy the positivity. An example is given as follows. Let $x, y \in X$ with $x \neq y$. Define $u \in P^1_{USCB}(X)$ given by $\langle u\rangle_\alpha = \{x\}$ for all $\alpha \in [0,1]$, and $v \in P^1_{USCB}(X)$ given by $\langle v\rangle_\alpha = \{x\}$ for $\alpha \in (0,1]$ and $\langle v\rangle_0 = \{x,y\}$. Then $u \neq v$ and $H_{\rm end}(u,v) = 0$.

Consider the function $f : F^1_{USC}(X) \to P^1_{USC}(X)$ given by $f(u) = \text{send}\, u$. Then

- f is an isometric embedding of $(F^1_{USC}(X), H_{\text{send}})$ in $(P^1_{USC}(X), H_{\text{send}})$.
- $f|_{F^1_{USCB}(X)}$ is an isometric embedding of $(F^1_{USCB}(X), H_{\text{send}})$ in $(P^1_{USCB}(X), H_{\text{send}})$.

Hence $(F^1_{USC}(X), H_{\text{send}})$ can be embedded isometrically in $(P^1_{USC}(X), H_{\text{send}})$, and $(F^1_{USCB}(X), H_{\text{send}})$ can be embedded isometrically in $(P^1_{USCB}(X), H_{\text{send}})$.

For $u \in F^1_{USC}(X)$, we define $\overrightarrow{u} := f(u)$. Then $\overrightarrow{u} \in P^1_{USC}(X)$.

For $v \in P^1_{USC}(X)$, we define $v' \in f(F^1_{USC}(X)) \subseteq P^1_{USC}(X)$ given by

$$\langle v' \rangle_\alpha = \begin{cases} \langle v \rangle_\alpha, & \alpha \in (0,1], \\ \overline{\cup_{\alpha>0} \langle v \rangle_\alpha}, & \alpha = 0. \end{cases}$$

Define $\overleftarrow{v} := f^{-1}(v')$. Then $\overleftarrow{v} \in F^1_{USC}(X)$.

For $u, v \in F^1_{USC}(X)$, $H_{\text{send}}(u,v) = H_{\text{send}}(\overrightarrow{u}, \overrightarrow{v})$ and $H_{\text{end}}(u,v) = H_{\text{end}}(\overrightarrow{u}, \overrightarrow{v})$. For $u, v \in P^1_{USC}(X)$, $H_{\text{end}}(u,v) = H_{\text{end}}(\overleftarrow{u}, \overleftarrow{v})$.

For a subset U of $F^1_{USC}(X)$, we use $\overrightarrow{U}$ to denote the set $\{\overrightarrow{u} : u \in U\}$.

For $w \in P^1_{USC}(X)$, the following are equivalent: (i) $w \in \overrightarrow{F^1_{USC}(X)}$; (ii) $\langle w \rangle_0 = \overline{\bigcup_{\delta>0} \langle w \rangle_\delta}$; (iii) $w = w' = \overleftrightarrow{w}$; (iv) $\langle w \rangle_0 = [\overleftarrow{w}]_0$.

Theorem 3.1. *Let (X,d) be a metric space. For $u, v \in P^1_{USC}(X)$:*
(i) $H_{\text{end}}(u,v) \le H_{\text{send}}(u,v)$*;*
(ii) $H(\langle u \rangle_0, \langle v \rangle_0) \le H_{\text{send}}(u,v)$*;*
(iii) If $H_{\text{end}}(u,v) < 1$*, then* $H_{\text{send}}(u,v) \le H_{\text{end}}(u,v) + H(\langle u \rangle_0, \langle v \rangle_0)$*.*
For a sequence $\{u_n\}$ in $P^1_{USC}(X)$ and u in $P^1_{USC}(X)$:
(iv) $H_{\text{send}}(u_n, u) \to 0$ *if and only if* $H_{\text{end}}(u_n, u) \to 0$ *and* $H(\langle u_n \rangle_0, \langle u \rangle_0) \to 0$*;*
(v) $\lim^{(K)}_{n\to\infty} u_n = u$ *if and only if* $\lim^{(K)}_{n\to\infty} \underline{u_n} = \underline{u}$ *and* $\lim^{(K)}_{n\to\infty} \langle u_n \rangle_0 = \langle u \rangle_0$*.*

Proof. Clearly (i) and (ii) are true. The proof of (i) and (ii) are routine.

To show (i), let $(x,\alpha) \in \underline{u}$. If $\alpha = 0$, then $\overline{d}((x,\alpha), \underline{v}) = 0 \le H^*(u,v)$. If $\alpha > 0$, then $(x,\alpha) \in u$ and $\overline{d}((x,\alpha), \underline{v}) \le H^*(u,v)$. From the arbitrariness of (x,α) in $\underline{u}$, we have $H^*(\underline{u}, \underline{v}) = \sup_{(x,\alpha)\in\underline{u}} \overline{d}((x,\alpha), \underline{v}) \le H^*(u,v)$. Similarly $H^*(\underline{v}, \underline{u}) \le H^*(v,u)$. So $H_{\text{end}}(u,v) = H(\underline{u}, \underline{v}) \le H(u,v) = H_{\text{send}}(u,v)$, i.e. (i) is true.

To show (ii), let $(x,0) \in u$. If $(y,\alpha) \in v$, then $(y,0) \in v$ and $\overline{d}((x,0),(y,\alpha)) \ge \overline{d}((x,0),(y,0))$. Hence $\overline{d}((x,0), v) = \inf_{(y,\alpha)\in v} \overline{d}((x,0),(y,\alpha)) = \inf_{(y,0)\in v} \overline{d}((x,0),(y,0)) = \inf_{y \in \langle v \rangle_0} d(x,y) = d(x, \langle v \rangle_0)$. Since $(x,0) \in u$ if and only if $x \in \langle u \rangle_0$, thus

$H^*(\langle u\rangle_0, \langle v\rangle_0) = \sup_{x\in\langle u\rangle_0} d(x, \langle v\rangle_0) = \sup_{(x,0)\in u} \overline{d}((x,0), v) \le H^*(u,v)$. Similarly $H^*(\langle v\rangle_0, \langle u\rangle_0) \le H^*(v,u)$. So $H(\langle u\rangle_0, \langle v\rangle_0) \le H(u,v) = H_{\text{send}}(u,v)$, i.e. (ii) is true.

To show (iii), let $(x,\alpha) \in u$. Then $d((x,\alpha), v) \le \alpha + d(x, \langle v\rangle_0)$. If $\alpha > d((x,\alpha), \underline{v})$, then $d((x,\alpha), v) = d((x,\alpha), \underline{v})$. Hence

$$d((x,\alpha), v) \le \begin{cases} H_{\text{end}}(u,v), & \alpha > H_{\text{end}}(u,v), \\ \alpha + H(\langle u\rangle_0, \langle v\rangle_0), & \alpha \in [0,1]. \end{cases}$$

Thus for $u, v \in P^1_{USC}(X)$ with $H_{\text{end}}(u,v) < 1$

$$H_{\text{send}}(u,v) \le H_{\text{end}}(u,v) + H(\langle u\rangle_0, \langle v\rangle_0). \tag{3}$$

So (iii) is true. We can see that if one of the following conditions (i) $H_{\text{end}}(u,v) = 0$, (ii) $H(\langle u\rangle_0, \langle v\rangle_0) = 0$, and (iii) $H(\langle u\rangle_0, \langle v\rangle_0) = +\infty$ hold, then "=" can be obtained in (3). However, the converse is false.

(iv) follows immediately from (i), (ii) and (iii). Below we verify (v).

Suppose that $\lim^{(K)}_{n\to\infty} u_n = u$. To show $\lim^{(K)}_{n\to\infty} \underline{u_n} = \underline{u}$ and $\lim^{(K)}_{n\to\infty} \langle u_n\rangle_0 = \langle u\rangle_0$, we only need to show that

$$\underline{u} \subseteq \liminf_{n\to\infty} \underline{u_n}, \ \limsup_{n\to\infty} \underline{u_n} \subseteq \underline{u},$$
$$\langle u\rangle_0 \subseteq \liminf_{n\to\infty} \langle u_n\rangle_0, \ \limsup_{n\to\infty} \langle u_n\rangle_0 \subseteq \langle u\rangle_0.$$

Let $(x,\alpha) \in \underline{u}$. If $\alpha = 0$, then clearly $(x,\alpha) \in \liminf_{n\to\infty} \underline{u_n}$. If $\alpha > 0$, then $(x,\alpha) \in u$, and thus $(x,\alpha) \in \liminf_{n\to\infty} u_n \subseteq \liminf_{n\to\infty} \underline{u_n}$. So $\underline{u} \subseteq \liminf_{n\to\infty} \underline{u_n}$.

Let $(x,\alpha) \in \limsup_{n\to\infty} \underline{u_n}$. If $\alpha = 0$, then clearly $(x,\alpha) \in \underline{u}$. If $\alpha > 0$, then $(x,\alpha) \in \limsup_{n\to\infty} u_n = u \subseteq \underline{u}$. So $\limsup_{n\to\infty} \underline{u_n} \subseteq \underline{u}$.

Let $x \in \langle u\rangle_0$. Then $(x,0) \in u = \liminf_{n\to\infty} u_n$. Thus there is a sequence $\{(x_n, \alpha_n)\}$ such that $(x_n, \alpha_n) \in u_n$, $n = 1, 2, \ldots$ and $(x,0) = \lim_{n\to\infty}(x_n, \alpha_n)$. Hence $x_n \in \langle u_n\rangle_0$ and $x = \lim_{n\to\infty} x_n$. So $\langle u\rangle_0 \subseteq \liminf_{n\to\infty}\langle u_n\rangle_0$.

Let $x \in \limsup_{n\to\infty}\langle u_n\rangle_0$. Then there is a sequence $\{x_{n_i}\}$ such that $x_{n_i} \in \langle u_{n_i}\rangle_0$, $i = 1, 2, \ldots$ and $x = \lim_{i\to\infty} x_{n_i}$. Thus $(x,0) = \lim_{i\to\infty}(x_{n_i}, 0) \in \limsup_{n\to\infty} u_n = u$. Hence $x \in \langle u\rangle_0$. So $\limsup_{n\to\infty}\langle u_n\rangle_0 \subseteq \langle u\rangle_0$.

Suppose that $\lim^{(K)}_{n\to\infty} \underline{u_n} = \underline{u}$ and $\lim^{(K)}_{n\to\infty}\langle u_n\rangle_0 = \langle u\rangle_0$. To show $\lim^{(K)}_{n\to\infty} u_n = u$, we only need to show that

$$u \subseteq \liminf_{n\to\infty} u_n, \ \limsup_{n\to\infty} u_n \subseteq u.$$

Let $(x, \alpha) \in u$. If $\alpha = 0$, then $x \in \langle u \rangle_0 = \lim^{(K)}_{n\to\infty} \langle u_n \rangle_0$. Thus there is a sequence $\{x_n\}$ such that $x_n \in \langle u_n \rangle_0$, $n = 1, 2, \ldots$ and $x = \lim_{n\to\infty} x_n$. Hence $(x, \alpha) = (x, 0) = \lim_{n\to\infty}(x_n, 0) \in \liminf_{n\to\infty} u_n$.

If $\alpha > 0$, then from $(x, \alpha) \in \underline{u} = \lim^{(K)}_{n\to\infty} \underline{u_n}$, there is a sequence $\{(x_n, \alpha_n)\}$ such that $(x, \alpha) = \lim_{n\to\infty}(x_n, \alpha_n)$, and $(x_n, \alpha_n) \in \underline{u_n}$ and $\alpha_n > 0$ for $n = 1, 2, \ldots$. Thus $(x_n, \alpha_n) \in u_n$, $n = 1, 2, \ldots$ and hence $(x, \alpha) \in \liminf_{n\to\infty} u_n$.

Let $(x, \alpha) \in \limsup_{n\to\infty} u_n$. Then $(x, \alpha) \in \limsup_{n\to\infty} \underline{u_n} = \underline{u}$. Note that $x \in \limsup_{n\to\infty} \langle u_n \rangle_0 = \langle u \rangle_0$, thus $(x, \alpha) \in u$.

□

Let $u, v \in F^1_{USC}(X)$. Then $\overrightarrow{u}, \overrightarrow{v} \in P^1_{USC}(X)$, and the following holds

$$H_{\text{send}}(u, v) = H_{\text{send}}(\overrightarrow{u}, \overrightarrow{v}),$$
$$H_{\text{end}}(u, v) = H_{\text{end}}(\overrightarrow{u}, \overrightarrow{v}),$$
$$H([u]_0, [v]_0) = H(\langle \overrightarrow{u} \rangle_0, \langle \overrightarrow{v} \rangle_0).$$

Thus (i), (ii) and (iii) in Theorem 3.1 imply that

$$H_{\text{end}}(u, v) \le H_{\text{send}}(u, v), \tag{4}$$
$$H([u]_0, [v]_0) \le H_{\text{send}}(u, v), \tag{5}$$

and if $H_{\text{end}}(u, v) < 1$, then

$$H_{\text{send}}(u, v) \le H_{\text{end}}(u, v) + H([u]_0, [v]_0). \tag{6}$$

Proposition 3.2. *Given u, u_n, $n = 1, 2, \ldots$ in $F^1_{USC}(X)$. Then*

(i) $H_{\text{send}}(u_n, u) \to 0$ *if and only if* $H_{\text{end}}(u_n, u) \to 0$ *and* $H([u_n]_0, [u]_0) \to 0$.

(ii) $\lim^{(K)}_{n\to\infty} \text{send}\, u_n = \text{send}\, u$ *if and only if* $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ *and* $\lim^{(K)}_{n\to\infty} [u_n]_0 = [u]_0$.

Proof. (i) follows immediately from (4), (5) and (6). (i) can be seen as a special case of the clause (iv) in Theorem 3.1. (ii) can be seen as a special case of the clause (v) in Theorem 3.1.

□

We suspect that the following Theorem 3.3 is an already known conclusion, however we can't find this conclusion in the references that we can obtain. It can be proved in a similar fashion to Theorem 4.1 in [10]. In this paper, we exclude the case that $C = \emptyset$.

Theorem 3.3. *[10] Suppose that C, C_n are sets in $C(X)$, $n = 1, 2, \ldots$. Then $H(C_n, C) \to 0$ implies that $\lim^{(K)}_{n\to\infty} C_n = C$.*

Remark 3.4. From Theorem 3.3, we obtain that for u, u_n, $n = 1, 2, \ldots$ in $P^1_{USC}(X)$: (i) $H_{\rm end}(u_n, u) \to 0$ implies that $\lim^{(K)}_{n\to\infty} \underline{u_n} = \underline{u}$; (ii) $H_{\rm send}(u_n, u) \to 0$ implies that $\lim^{(K)}_{n\to\infty} u_n = u$.

So for u, u_n, $n = 1, 2, \ldots$ in $F^1_{USC}(X)$: (iii) $H_{\rm end}(u_n, u) \to 0$ implies that $\lim^{(\Gamma)}_{n\to\infty} u_n = u$; (iv) $H_{\rm send}(u_n, u) \to 0$ implies that $\lim^{(K)}_{n\to\infty} \operatorname{send} u_n = \operatorname{send} u$.

The converses of the implications in clauses (i), (ii), (iii) and (iv) in Remark 3.4 are false. Let $u = [0, +\infty)_{F(\mathbb{R})}$ and for $n = 1, 2, \ldots$, let $u_n = [0, n]_{F(\mathbb{R})}$. Then $\lim^{(K)}_{n\to\infty} \operatorname{send} u_n = \operatorname{send} u$, but $H_{\rm end}(u_n, u) = 1 \not\to 0$. So combined with Proposition 3.2, the converses of the implications in (iii) and (iv) are false, and thus the converses of the implications in (i) and (ii) are false.

4. Level characterizations of Γ-convergence

In this section, we investigate the level characterizations of the Γ-convergence. It is shown that under some conditions, the Γ-convergence of fuzzy sets can be decomposed to the Kuratowski convergence of certain α-cuts.

We need the following conclusion, which is Lemma 2.1 in [10].

Lemma 4.1. *[10] Let (X, d) be a metric space, and C_n, $n = 1, 2, \ldots$, be a sequence of sets in X. Suppose that $x \in X$. Then*
(i) $x \in \liminf_{n\to\infty} C_n$ if and only if $\lim_{n\to\infty} d(x, C_n) = 0$,
(ii) $x \in \limsup_{n\to\infty} C_n$ if and only if there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $\lim_{k\to\infty} d(x, C_{n_k}) = 0$.

Rojas-Medar and Román-Flores [18] have introduced the following useful property of the Γ-convergence.

Theorem 4.2. *[18] Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F^1_{USC}(X)$. Then $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ if and only if for all $\alpha \in (0, 1]$,*

$$\{u > \alpha\} \subseteq \liminf_{n\to\infty} [u_n]_\alpha \subseteq \limsup_{n\to\infty} [u_n]_\alpha \subseteq [u]_\alpha. \tag{7}$$

Proof. Sufficiency. Suppose that (7) is true for all $\alpha \in (0, 1]$. To show that $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, it suffices to prove that $\operatorname{end} u \subseteq \liminf_{n\to\infty} \operatorname{end} u_n$ and $\limsup_{n\to\infty} \operatorname{end} u_n \subseteq \operatorname{end} u$.

To show that $\mathrm{end}\, u \subseteq \liminf_{n\to\infty} \mathrm{end}\, u_n$, let $(x, \alpha) \in \mathrm{end}\, u$. We only need to show that $(x, \alpha) \in \liminf_{n\to\infty} \mathrm{end}\, u_n$. It suffices to verify, by Lemma 4.1, that

$$\lim_{n\to\infty} \overline{d}((x, \alpha), \mathrm{end}\, u_n) = 0. \tag{8}$$

If $\alpha = 0$, then clearly (8) is true. Suppose $\alpha \in (0, 1]$. Then for each $k \in \mathbb{N}$, $x \in \{u > \alpha(1 - \frac{1}{2k})\}$, and therefore from (7), $x \in \liminf_{n\to\infty}[u_n]_{\alpha(1-\frac{1}{2k})}$. By Lemma 4.1, $\lim_{n\to\infty} d(x, [u_n]_{\alpha(1-\frac{1}{2k})}) = 0$. So there is an N such that $d(x, [u_n]_{\alpha(1-\frac{1}{2k})}) < \frac{1}{2k}$ for all $n \geq N$. Hence $\overline{d}((x, \alpha), \mathrm{end}\, u_n) < \frac{1}{k}$ for all $n \geq N$. From the arbitrariness of k, we thus have that (8) is true.

To show that $\limsup_{n\to\infty} \mathrm{end}\, u_n \subseteq \mathrm{end}\, u$, let $(x, \alpha) \in \limsup_{n\to\infty} \mathrm{end}\, u_n$. it suffices to verify that $(x, \alpha) \in \mathrm{end}\, u$. If $\alpha = 0$, then clearly $(x, \alpha) \in \mathrm{end}\, u$. Suppose that $\alpha > 0$. Then there is an sequence $\{(x_{n_k}, \alpha_{n_k})\}_{k=1}^{+\infty}$ satisfying $(x_{n_k}, \alpha_{n_k}) \in \mathrm{end}\, u_{n_k}$ for $k = 1, 2, \ldots$ and $\lim_{k\to\infty} \overline{d}(x_{n_k}, \alpha_{n_k}), (x, \alpha)) = 0$. So for each $m \in \mathbb{N}$, $x \in \limsup_{n\to\infty}[u_n]_{\alpha(1-\frac{1}{m})}$, and hence, by (7), $x \in [u]_{\alpha(1-\frac{1}{m})}$. From the arbitrariness of m, we have $x \in [u]_\alpha$, and thus $(x, \alpha) \in \mathrm{end}\, u$.

Necessity. Suppose that $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. Let $\alpha \in (0, 1]$. To show (7), we only need to verify that $\{u > \alpha\} \subseteq \liminf_{n\to\infty}[u_n]_\alpha$ and $\limsup_{n\to\infty}[u_n]_\alpha \subseteq [u]_\alpha$.

Suppose that $x \in \{u > \alpha\}$. Then there is a $\beta > \alpha$ such that $(x, \beta) \in \mathrm{end}\, u = \liminf_{n\to\infty} \mathrm{end}\, u_n$. Hence there is a sequence $\{(x_n, \beta_n)\}_{n=1}^{+\infty}$ satisfying $(x_n, \beta_n) \in \mathrm{end}\, u_n$, $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \overline{d}((x_n, \beta_n), (x, \beta)) = 0$. Notice that there is an $N \in \mathbb{N}$ such that $x_n \in \{u_n > \alpha\}$ for all $n \geq N$. So $x \in \liminf_{n\to\infty}[u_n]_\alpha$. Thus we have $\{u > \alpha\} \subseteq \liminf_{n\to\infty}[u_n]_\alpha$.

Suppose that $x \in \limsup_{n\to\infty}[u_n]_\alpha$. Then there is an sequence $\{x_{n_j}\}_{j=1}^{+\infty}$ satisfying $x_{n_j} \in [u_{n_j}]_\alpha$, $j = 1, 2, \ldots$ and $\lim_{j\to\infty} d(x_{n_j}, x) = 0$. Hence $\lim_{j\to\infty} \overline{d}((x_{n_j}, \alpha), (x, \alpha)) = 0$. Notice $(x_{n_j}, \alpha) \in \mathrm{end}\, u_{n_j}$ and therefore $(x, \alpha) \in \limsup_{n\to\infty} \mathrm{end}\, u_n = \mathrm{end}\, u$. So $x \in [u]_\alpha$. Thus we have $\limsup_{n\to\infty}[u_n]_\alpha \subseteq [u]_\alpha$.

□

Remark 4.3. Rojas-Medar and Román-Flores (Proposition 3.5 in [18]) presented the statement in Theorem 4.2 when u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in E^m. Since we can't find a proof for Proposition 3.5 in [18], we give a proof here.

Corollary 3.2.13 in [15] states that for each net of subsets $\{A_n, n \in D\}$ in a topological space, $\liminf C_n$ and $\limsup C_n$ according to Definition 3.1.4 in

[15] are closed sets. From the fact illustrated in Remark 2.3, we know that Corollary 3.2.13 in [15] implies the following Theorem 4.4.

Theorem 4.4 is Theorem 2.1 in [10]. Of course, the conclusion that $\limsup_{n\to\infty} C_n$ are closed sets in (X, d) in Theorem 4.4 can also be deduced from the fact that $\limsup_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m\geq n} C_m}$.

Theorem 4.4. *[10, 15] Let (X, d) be a metric space and let $\{C_n\}$ be a sequence of sets in X. Then $\liminf_{n\to\infty} C_n$ and $\limsup_{n\to\infty} C_n$ are closed sets in (X, d).*

Theorem 4.5. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F^1_{USC}(X)$. Then $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ if and only if for all $\alpha \in (0, 1]$,*

$$\overline{\{u > \alpha\}} \subseteq \liminf_{n\to\infty} [u_n]_\alpha \subseteq \limsup_{n\to\infty} [u_n]_\alpha \subseteq [u]_\alpha.$$

Proof. The desired result follows from Theorems 4.2 and 4.4. □

Remark 4.6. Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F^1_{USC}(X)$. If $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, then by Theorems 4.4 and 4.5, $[u]_0 = \overline{\{u > 0\}} \subseteq \liminf_{n\to\infty}[u_n]_0$. In this case $[u]_0 \subsetneqq \liminf_{n\to\infty}[u_n]_0$ could happen. For example, let $u = \widehat{0}_{F(\mathbb{R})}$ and for $n = 1, 2, \ldots$, define $u_n \in F^1_{USCB}(\mathbb{R})$ as

$$u_n = \begin{cases} 1, & x = 0, \\ 1/n, & x \in (0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Then $H_{\rm end}(u_n, u) \to 0$, and therefore from Remark 3.4 $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. And $[u]_0 = \{0\} \subsetneqq [0, 1] = \liminf_{n\to\infty}[u_n]_0$.

By Proposition 3.2, $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $[u]_0 \supseteq \limsup_{n\to\infty}[u_n]_0$ if and only if $\lim^{(K)}_{n\to\infty} \text{send}\, u_n = \text{send}\, u$.

Let $u \in F(X)$. Denote

$$D(u) := \{\alpha \in (0, 1) : [u]_\alpha \nsubseteq \overline{\{u > \alpha\}}\},$$
$$P(u) := \{\alpha \in (0, 1) : \overline{\{u > \alpha\}} \subsetneqq [u]_\alpha\}.$$

A number α in $P(u)$ is called a platform point of u. Clearly $P(u) \subseteq D(u)$. $P(u) \subsetneqq D(u)$ could happen. See Example 4.7.

Example 4.7. let $u \in F(\mathbb{R})$ defined by

$$u(x) = \begin{cases} 1, & x \in (0,1), \\ 0.6, & x \in [1,3], \\ 0, & x \in \mathbb{R} \setminus (0,3]. \end{cases}$$

Then $P(u) = \emptyset$ and $D(u) = \{0.6\}$. So $P(u) \subsetneqq D(u)$.

Theorem 5.1 in [10] says that $D(u)$ is at most countable when $u \in F(\mathbb{R}^m)$. In a similar manner, we can show the following Theorem 4.8. Then we will give Theorem 4.9 which is a generalization of the above conclusions. Its proof is based on the well-known result:

- each separable metric space is homeomorphic to a subspace of the **Hilbert space** $\boldsymbol{l^2} := \{(x_i)_{i=1}^{+\infty} : \sum_{i=1}^{+\infty} x_i^2 < +\infty\}$.

Theorem 4.8. *Let $u \in F(l^2)$. Then the set $D(u)$ is at most countable.*

Proof. The proof is similar to that of Theorem 5.1 in [10]. A sketch of the proof is given below.

Similarly as in [10], for $u \in F(l^2)$, $t \in l^2$ and $r \in \mathbb{R}^+$, we can define $S_{u,t,r}(\cdot,\cdot) : \mathbf{S}^1 \times [0,1] \to \{-\infty\} \cup \mathbb{R}$ by

$$S_{u,t,r}(e,\alpha) = \begin{cases} -\infty, & \text{if } [u]_\alpha \cap \overline{B(t,r)} = \emptyset, \\ \sup\{\langle e, x-t\rangle : x \in [u]_\alpha \cap \overline{B(t,r)}\}, & \text{if } [u]_\alpha \cap \overline{B(t,r)} \neq \emptyset, \end{cases}$$

where $\mathbf{S}^1 := \{e \in l^2 : \|e\| = 1\}$ and $\overline{B(t,r)} := \{x \in l^2 : \|x-t\| \le r\}$.

Similarly as in [10], we can define $D(u,t,r,e)$, which is the discontinuous point of $S_{u,t,r}(e,\cdot)$.

Proceed as in the proof of Lemma A.1. in [10], we can show the conclusion corresponding to Lemma A.1. in [10]: $D(u,t,r) = \cup_{e\in\mathbf{S}^1} D(u,t,r,e)$ is at most countable.

Here we mention that the narrative of the proof of formula (A.6) in Theorem 5.1 in [10] can be slightly simplified. The detailed operations are performed as follows: replace Lines 1 and 2 from the bottom in Page 82 and Lines 1 and 2 in Page 83 in [10] by

- Since $2\langle a,b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ for each $a, b \in \mathbb{R}^m$, then

$$\langle e, x-q\rangle = \frac{\langle y-q, x-q\rangle}{\|y-q\|} = \frac{\|x-q\|^2 + \|y-q\|^2 - \|x-y\|^2}{2\|y-q\|}$$

Note that $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ for each $a, b \in l^2$. So proceed as in the proof of Theorem 5.1 in [10], we obtain the conclusion corresponding to Theorem 5.1 in [10]: $D(u)$ is at most countable.

□

Theorem 4.9. *Let (X, d) be a metric space and $u \in F(X)$. If $([u]_0, d)$ is separable, then the set $D(u)$ is at most countable.*

Proof. Let f be a homeomorphism from $([u]_0, d)$ to a subspace of l^2. Consider $u_f \in F(l^2)$ defined by

$$u_f(t) = \begin{cases} u(f^{-1}(t)), & t \in f([u]_0), \\ 0, & t \in l^2 \setminus f([u]_0). \end{cases}$$

Then by Theorem 4.8, $D(u_f)$ is at most countable. To show that $D(u)$ is at most countable, it suffices to show that $D(u) = D(u_f)$.

For $S \subseteq [u]_0$, let $\overline{S}^{[u]_0}$ denote the topological closure of S in $([u]_0, d)$. For $W \subseteq f([u]_0)$, let $\overline{W}^{l^2}$ denote the topological closure of W in l^2, and let $\overline{W}^{f([u]_0)}$ denote the topological closure of W in $f([u]_0)$, here we see $f([u]_0)$ as a metric subspace of l^2.

Note that for each $x \in [u]_0$, $u(x) = u_f(f(x))$. So for each $\alpha \in [0, 1]$ and each $x \in [u]_0$,

$$x \in [u]_\alpha \Leftrightarrow f(x) \in [u_f]_\alpha,$$

and

$$x \notin \overline{\{u > \alpha\}} \Leftrightarrow x \notin \overline{\{u > \alpha\}}^{[u]_0} \Leftrightarrow f(x) \notin \overline{\{u_f > \alpha\}}^{f([u]_0)} \Leftrightarrow f(x) \notin \overline{\{u_f > \alpha\}}^{l^2}.$$

This implies $D(u) = D(u_f)$, since
$D(u) = \{\alpha \in (0, 1) :$ there exists $x \in [u]_\alpha$ such that $x \notin \overline{\{u > \alpha\}}\}$, and
$D(u_f) = \{\alpha \in (0, 1) :$ there exists $f(x) \in [u_f]_\alpha$ such that $f(x) \notin \overline{\{u_f > \alpha\}}^{l^2}\}$.

□

Corollary 4.10. *Let (X, d) be a separable metric space and $u \in F(X)$. Then the set $D(u)$ is at most countable.*

Proof. Since every subspace of a separable metric space is separable, $([u]_0, d)$ is separable. Thus, by Theorem 4.9, $D(u)$ is at most countable.

□

Remark 4.11. It is well-known that both $\mathbb{R}^m$ and l^2 are separable metric space. Thus both Theorem 5.1 in [10] and Theorem 4.8 are special cases of Corollary 4.10, which is a corollary of Theorem 4.9. So Theorem 4.9 is a generalization of Theorem 5.1 in [10] and Theorem 4.8.

Remark 4.12. Theorems 4.8, 4.9 and Corollary 4.10 remain true if $D(u)$ is replaced by $P(u)$, since $P(u) \subseteq D(u)$.

If $u \in F^1_{USC}(X)$, then clearly $P(u) = D(u)$.

Theorem 4.13. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F^1_{USC}(X)$. Then the following statements are true.*

(i) If there is a dense set P in $(0, 1)$ such that $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$ for $\alpha \in P$, then $u = \lim^{(\Gamma)}_{n\to\infty} u_n$.

(ii) If $u = \lim^{(\Gamma)}_{n\to\infty} u_n$, then $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$ for all $\alpha \in (0, 1) \setminus P(u)$.

Proof. The proof of (i) is similar to "(ii) $\Rightarrow$ (i)" in the proof of Theorem 6.2 in [10]. (ii) follows immediately from Theorem 4.5.

$\square$

The following theorem gives a condition under which the Γ-convergence of fuzzy sets can be decomposed to the Kuratowski convergence of certain α-cuts.

Theorem 4.14. *Suppose that u, u_n, $n = 1, 2, \ldots$, are fuzzy sets in $F^1_{USC}(X)$. If $([u]_0, d)$ is separable, then the following are equivalent:*

(i) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$;

(ii) $\lim^{(K)}_{n\to\infty} u_n = u$ holds a.e. on $\alpha \in (0, 1)$;

(iii) $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$ holds for all $\alpha \in (0, 1) \setminus P(u)$;

(iv) There is a dense subset P of $(0, 1)\setminus P(u)$ such that $\lim^{(K)}_{n\to\infty}[u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$;

(v) There is a countable dense subset P of $(0, 1)\setminus P(u)$ such that $\lim^{(K)}_{n\to\infty}[u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$.

Proof. (i)⇒(iii) is (ii) of Theorem 4.13. Clearly (iii)⇒(iv)⇒(v). (i) of Theorem 4.13 implies that (v)⇒(i) and (ii)⇒(i).

We shall complete the proof by showing that (iii)⇒(ii). This follows from the fact that $P(u)$ is at most countable, which is pointed out by Theorem 4.9 and Remark 4.12.

$\square$

Remark 4.15. By Lemma 5.3, if $u \in F^1_{USCG}(X)$ then $P(u)$ is at most countable. So from the proof of Theorem 4.14, we know that Theorem 4.14 remains true if "$([u]_0, d)$ is separable" is replaced by "$u \in F^1_{USCG}(X)$".

Here we mention that the condition "$u \in F^1_{USCG}(X)$" implies the condition "$([u]_0, d)$ is separable". Let $u \in F^1_{USCG}(X)$. Then for each $\alpha \in (0, 1]$, $([u]_\alpha, d)$ is separable, because each compact metric space is separable. Since $[u]_0 = \overline{\cup_{n=1}^{+\infty}[u]_{1/n}}$, we have that $([u]_0, d)$ is separable.

Remark 4.16. Example Appendix A.4 shows that there exist u and u_n, $n = 1, 2, \dots$ in $F^1_{USC}(\prod_{x\in(0,1]}[0, 3])$ such that
(i) $H_{\rm send}(u_n, u) \to 0$,
(ii) $(0, 1) \setminus P(u) = \emptyset$,
(iii) $\{[u_n]_\alpha\}$ does not Kuratowski converge to $[u]_\alpha$ when $\alpha \in (0, 1]$,
(iv) $u = \lim^{(\Gamma)}_{n\to\infty} u_n$.

The above (i)-(iii) are shown in Example Appendix A.4. From Proposition 3.2 and Remark 3.4, the $H_{\rm send}$ convergence implies the Γ-convergence on $F^1_{USC}(\prod_{x\in(0,1]}[0, 3])$. Hence (i) implies (iv).

From (ii), for each $\{v_n\}$ in $F^1_{USC}(\prod_{x\in(0,1]}[0, 3])$, the statement "$\{[v_n]_\alpha\}$ Kuratowski converges to $[u]_\alpha$ for $\alpha \in (0, 1) \setminus P(u)$ " is true. However "$u = \lim^{(\Gamma)}_{n\to\infty} v_n$" is not necessarily hold.

So from Example Appendix A.4 we know: the converse of the implication in statement (i) of Theorem 4.13 does not hold; the converse of the implication in statement (ii) of Theorem 4.13 does not hold; the condition "$([u]_0, d)$ is separable" can not be deleted in Theorem 4.14.

5. Level characterizations of endograph metric convergence

In this section, we discuss the level characterizations of endograph metric convergence. It is shown that under some condition, the $H_{\rm end}$ metric convergence of fuzzy sets can be decomposed to the Hausdorff metric convergence of certain α-cuts.

Let u be a fuzzy set in $F^1_{USC}(X)$. Denote

$$P_0(u) := \{\alpha \in (0,1) : \lim_{\beta\to\alpha} H([u]_\beta, [u]_\alpha) \neq 0\}.$$

Clearly $P(u) \subseteq P_0(u)$ for $u \in F^1_{USC}(X)$. $P(u) \subsetneqq P_0(u)$ could happen. See Example 5.1.

Example 5.1. Let $u \in F^1_{USC}(\mathbb{R}^2)$ defined by

$$[u]_\alpha = \{0\} \cup \{z : \arg z \in [\alpha, 1]\} \text{ for each } \alpha \in [0,1],$$

here we write each $(x,y) \in \mathbb{R}^2$ as a complex number $z = x + iy$. Then $P(u) = \emptyset$ and $P_0(u) = (0,1)$. So $P(u) \subsetneqq P_0(u)$.

This example also shows that for $u \in F^1_{USC}(X)$, $P_0(u)$ need not be at most countable even X is a separable metric space.

Lemma 5.2. *[11] Let $U_n \in K(X)$ for $n = 1, 2, \ldots$.*
(i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.
(ii) If $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq \ldots$ and $V = \overline{\bigcup_{n=1}^{+\infty} V_n} \in K(X)$, then $H(V_n, V) \to 0$ as $n \to +\infty$.

Proof. This is Lemma 4.4 in [11]. Here we give a proof using Theorem 2.2.

Since for $n = 1, 2, \ldots$, U_n is closed in X, then U is a closed subset of $U_1 \in K(X)$. Hence $U \in K(X)$. Since $U_n \in K(U_1)$ for $n = 1, 2, \ldots$, then by Theorem 2.2, $\{U_n\}$ has a subsequence which converges to $D \in K(U_1)$. Then clearly $H(U_n, D) \to 0$, and thus by Theorem 3.3, $D = \lim^{(K)}_{n\to\infty} U_n = U$. So (i) is true.

Since $V_n \in K(V)$ for $n = 1, 2, \ldots$, then by Theorem 2.2, $\{V_n\}$ has a subsequence which converges to $C \in K(V)$. Then clearly $H(V_n, C) \to 0$, and thus by Theorem 3.3, $C = \lim^{(K)}_{n\to\infty} V_n = V$. So (ii) is true.

In chinaXiv:202107.00011v3, which is an early version of this paper submitted in 2021-08-01, we gave this proof of (ii) and pointed out that (i) can be shown in a similar way.

$\square$

Lemma 5.3. *For $u \in F^1_{USCG}(X)$:*
(i) $\lim_{\beta\to\alpha-} H([u]_\beta, [u]_\alpha) = 0$ holds for $\alpha \in (0,1]$;
(ii) $\lim_{\gamma\to\alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$ holds for $\alpha \in (0,1)$;
(iii) $\lim_{\delta\to\alpha} H([u]_\delta, [u]_\alpha) = 0$ holds for $\alpha \in (0,1) \setminus P(u)$ and $P(u) = P_0(u)$;
(iv) $P_0(u)$ is at most countable.

Proof. Lemma 5.2 implies (i) and (ii). (i) and (ii) imply (iii). (iv) is Lemma 6.12 in [11].

□

Remark 5.4. (iv) in Lemma 5.3 can also be shown in such a way: let $u \in F^1_{USCG}(X)$, then by Remark 4.15, $([u]_0, d)$ is separable, and thus by Theorem 4.9 and Remark 4.12, $P(u)$ is countable. So from (iii) in Lemma 5.3, we obtain that $P_0(u)$ is at most countable.

Theorem 5.5. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$.*
(i) $\lim_{n\to\infty} H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) = 0$ *if and only if for each* $\alpha \in [0, 1)$ *and* $\xi \in (0, 1-\alpha]$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$.
(ii) $\lim_{n\to\infty} H^*(\mathrm{end}\, u_n, \mathrm{end}\, u) = 0$ *if and only if for each* $\alpha \in (0, 1]$ *and* $\zeta \in (0, \alpha]$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$.

Proof. We only prove (i). (ii) can be proved similarly.

Necessity. Assume that $\lim_{n\to\infty} H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) = 0$. Let $\alpha \in [0, 1)$ and $\xi \in (0, 1-\alpha]$. Then for each $\varepsilon \in (0, \xi)$, there exists an $N(\varepsilon)$ such that for all $n \geq N$,

$$H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) < \varepsilon,$$

and then

$$H^*([u]_{\alpha+\xi}, [u_n]_\alpha) < \varepsilon.$$

From the arbitrariness of ε in $(0, \xi)$, we have $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$.

Sufficiency. Let $\varepsilon > 0$. Select a $k \in \mathbb{N}$ with $2/k < \varepsilon$. From (i), we have that for $l = 2, \ldots, k$, $\lim_{n\to\infty} H^*([u]_{l/k}, [u_n]_{(l-1)/k}) = 0$. So there is an $N(\varepsilon)$ such that for all $n \geq N$ and $l = 2, \ldots, k$,

$$H^*([u]_{l/k}, [u_n]_{(l-1)/k}) < \varepsilon. \tag{9}$$

Let $(x, \alpha) \in \mathrm{end}\, u$. If $\alpha \leq \varepsilon$, then $d((x, \alpha), \mathrm{end}\, u_n) \leq \varepsilon$. Suppose $\alpha > \varepsilon$. Then we can choose $l \in \{2, \ldots, k-1\}$ such that $l/k < \alpha \leq (l+1)/k$. Hence by (9), for $n \geq N$,

$$\begin{aligned} & d((x, \alpha), \mathrm{end}\, u_n) \\ & \leq d(x, [u_n]_{(l-1)/k}) + 2/k \\ & < H^*([u]_{l/k}, [u_n]_{(l-1)/k}) + \varepsilon < 2\varepsilon. \end{aligned}$$

From the arbitrariness of $(x, \alpha) \in \mathrm{end}\, u$, it follows that $H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) < 2\varepsilon$ for all $n \geq N$.

Thus $\lim_{n\to\infty} H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) = 0$ from the arbitrariness of $\varepsilon > 0$.

□

Theorem 5.6. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$.*

(i) The following are equivalent:
(i-1) $\lim_{n\to\infty} H^*(\mathrm{end}\, u, \mathrm{end}\, u_n) = 0$*;*
(i-2) For each $\alpha \in [0, 1)$ *and* $\xi \in (0, 1 - \alpha]$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$*;*
(i-3) There is a dense subset P *of* $[0, 1)$ *such that for each* $\alpha \in P$ *and* $\xi \in (0, 1 - \alpha]$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$.

(ii) The following are equivalent:
(ii-1) $\lim_{n\to\infty} H^*(\mathrm{end}\, u_n, \mathrm{end}\, u) = 0$*;*
(ii-2) for each $\alpha \in (0, 1]$ *and* $\zeta \in (0, \alpha]$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$*;*
(ii-3) There is a dense subset P *of* $(0, 1]$ *such that for each* $\alpha \in P$ *and* $\zeta \in (0, \alpha]$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$.

Proof. Let $\alpha \in [0, 1)$ and $\xi \in (0, 1 - \alpha]$. Choose $\beta \in P \cap [\alpha, \alpha + \xi)$. Then

$$H^*([u]_{\alpha+\xi}, [u_n]_\alpha) \le H^*([u]_{\alpha+\xi}, [u_n]_\beta).$$

Using this fact, we see that (i-3)⇒(i-2).

(i-2)⇒(i-3) is obvious. From Theorem 5.5, we have (i-1)⇔(i-2). Thus (i) is proved. Similarly, we can prove (ii).

□

Remark 5.7. Clearly, (i-2) is equivalent to the following (i-2)′, and (ii-2) is equivalent to the following (ii-2)′:
(i-2)′ For each $\alpha \in [0, 1)$ and each sequence $\{\xi_m\}$ with $\xi_m \le 1 - \alpha$ and $\xi_m \to 0+$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi_m}, [u_n]_\alpha) = 0$;
(ii-2)′ For each $\alpha \in (0, 1]$ and each sequence $\{\zeta_m\}$ with $\zeta_m \le \alpha$ and $\zeta_m \to 0+$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta_m}) = 0$.

Similarly, we can give (i-3)′ and (ii-3)′ which are equivalent to (i-3) and (ii-3), respectively.

Corollary 5.8. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. Then the following are equivalent:*
(i) $\lim_{n\to\infty} H_{\mathrm{end}}(u_n, u) = 0$*;*
(ii) For each $\alpha \in (0, 1)$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$ *when* $\xi \in (0, 1 - \alpha]$, *and* $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$ *when* $\zeta \in (0, \alpha]$*;*
(iii) There is a dense subset P *of* $(0, 1)$ *such that for each* $\alpha \in P$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$ *when* $\xi \in (0, 1 - \alpha]$, *and* $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$ *when* $\zeta \in (0, \alpha]$.

Proof. The desired result follows from Theorem 5.6. The proof is routine.

Note that $\lim_{n\to\infty} H_{\rm end}(u_n, u) = 0$ if and only if $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$ and $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$. So Theorem 5.6 implies that (i)⇒(ii) and (iii)⇒(i). Clearly (ii)⇒(iii). So (i)⇔(ii)⇔(iii).

□

Lemma 5.9. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$, and let P be a dense subset of $(0, 1)$.*
(i) If for each $\alpha \in P$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$, then $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$;
(ii) If for each $\alpha \in P$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$, then $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$;
(iii) If for each $\alpha \in P$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ and $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$, then $\lim_{n\to\infty} H_{\rm end}(u_n, u) = 0$.

Proof. The desired results follow from Theorem 5.6. The proof is routine.

Clearly for each $\alpha \in [0, 1)$, $\overline{\{u > \alpha\}} \supseteq [u]_{\alpha+\xi}$ when $\xi \in (0, 1-\alpha]$. Thus the assumption for each $\alpha \in P$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ implies for each $\alpha \in P$ and $\xi \in (0, 1-\alpha]$, $\lim_{n\to\infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$. Then u, u_n, $n = 1, 2, \ldots$ satisfy the condition (i-3) in Theorem 5.6. Hence by Theorem 5.6, $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$. So (i) is true.

Clearly for each $\alpha \in (0, 1]$, $[u]_\alpha \subseteq [u]_{\alpha-\zeta}$ when $\zeta \in (0, \alpha]$. Thus the assumption for each $\alpha \in P$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$ implies that for each $\alpha \in P$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$ when $\zeta \in (0, \alpha]$. Then u, u_n, $n = 1, 2, \ldots$ satisfy the condition (ii-3) in Theorem 5.6. Hence by Theorem 5.6, $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$. So (ii) is true.

(iii) follows immediately from (i) and (ii) since $\lim_{n\to\infty} H_{\rm end}(u_n, u) = 0$ if and only if $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$ and $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$.

We can see that (iii) follows immediately from Corollary 5.8.

□

Lemma 5.10. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$.*
(i) Let $\alpha \in [0, 1)$. If $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$, and $\lim_{\gamma\to\alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$, then $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$.
(ii) Let $\alpha \in (0, 1]$. If $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$, and $\lim_{\beta\to\alpha-} H([u]_\alpha, [u]_\beta) = 0$, then $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$.

Proof. We only prove (i). (ii) can be proved similarly.

Let $\varepsilon > 0$. Since $\lim_{\gamma\to\alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$, then there is a $\gamma(\alpha) \in (\alpha, 1]$ such that $H(\overline{\{u > \alpha\}}, [u]_\gamma) < \varepsilon/2$. By Theorem 5.5 (i), $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) =$

0 implies that $\lim_{n\to\infty} H^*([u]_\gamma, [u_n]_\alpha) = 0$. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $H^*([u]_\gamma, [u_n]_\alpha) < \varepsilon/2$. Hence for all $n \geq N$,

$$H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) \leq H(\overline{\{u > \alpha\}}, [u]_\gamma) + H^*([u]_\gamma, [u_n]_\alpha) < \varepsilon.$$

From the arbitrariness of $\varepsilon > 0$, we thus have $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$. $\square$

The assumption that $\lim_{\gamma\to\alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$ in (i) of Lemma 5.10 can not be omitted. The assumption that $\lim_{\beta\to\alpha-} H([u]_\alpha, [u]_\beta) = 0$ in (ii) of Lemma 5.10 also can not be omitted. The following Examples 5.11 and 5.12 are counterexamples.

Example 5.11. Let

$$[u]_\alpha = \begin{cases} (-\infty, \frac{\frac{2}{3}}{\alpha - \frac{1}{3}}], & \alpha \in (\frac{1}{3}, 1], \\ (-\infty, +\infty), & \alpha \in [0, \frac{1}{3}], \end{cases}$$

and let

$$[u_n]_\alpha = \begin{cases} (-\infty, \frac{1-\frac{1}{3}\frac{n-1}{n}}{\alpha - \frac{1}{3}\frac{n-1}{n}}], & \alpha \in (\frac{1}{3}\frac{n-1}{n}, 1], \\ (-\infty, +\infty), & \alpha \in [0, \frac{1}{3}\frac{n-1}{n}], \end{cases} \quad n = 1, 2, \ldots$$

Then u and u_n, $n = 1, 2, \ldots$ are fuzzy sets in $F^1_{USC}(\mathbb{R})$, and $\lim_{\gamma\to\frac{1}{3}+} H([u]_\gamma, \overline{\{u > \frac{1}{3}\}}) = +\infty \not\to 0$.

It can be seen that $H_{\rm end}(u, u_n) \to 0$. However

$$H^*(\overline{\{u > \frac{1}{3}\}}, [u_n]_{\frac{1}{3}}) = H^*([u]_{\frac{1}{3}}, [u_n]_{\frac{1}{3}}) = H^*((-\infty, +\infty),\ (-\infty, \frac{3 - \frac{n-1}{n}}{1 - \frac{n-1}{n}}]) = +\infty \not\to 0.$$

Example 5.12. Let

$$[u]_\alpha = \begin{cases} \{1\}, & \alpha = 1, \\ \{1\} \cup (-\infty, -\frac{1}{1-\alpha}], & \alpha \in [0, 1), \end{cases}$$

and let

$$[u_n]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in [0, 1 - \frac{1}{n}], \\ [u]_{1-\frac{1}{n}}, & \alpha \in [1 - \frac{1}{n}, 1], \end{cases} \quad n = 1, 2, \ldots.$$

Then u and u_n, $n = 1, 2, \ldots$ are fuzzy sets in $F^1_{USC}(\mathbb{R})$, and $\lim_{\beta\to 1-} H([u]_1, [u]_\beta) = +\infty \not\to 0$.

We can see that $H_{\rm end}(u, u_n) \to 0$. However

$$H^*([u_n]_1, [u]_1) = H^*(\{1\} \cup (-\infty, -n],\ \{1\}) = +\infty \not\to 0.$$

Theorem 5.13. *Let u be a fuzzy set in $F^1_{USCG}(X)$ and let u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$.*

(i) The following are equivalent:

(i-1) $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$;

(i-2) For each $\alpha \in (0, 1)$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$;

(i-3) There is a dense subset P of $(0, 1)$ such that for each $\alpha \in P$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$.

(ii) The following are equivalent:

(ii-1) $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$;

(ii-2) For each $\alpha \in (0, 1]$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$;

(ii-3) There is a dense subset P of $(0, 1]$ such that for each $\alpha \in P$, $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$.

Proof. The desired results follow from Lemma 5.3, Lemma 5.10 and Theorem 5.6. The proof is routine.

From Lemma 5.3 (ii), for $u \in F^1_{USCG}(X)$ and $\alpha \in (0, 1)$, $\lim_{\gamma\to\alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$. So by Lemma 5.10 (i), (i-1) implies (i-2). Clearly (i-2) implies (i-3). We shall complete the proof of (i) by showing that (i-3)⇒(i-1). This follows from Lemma 5.9 (i).

Similarly, we can show (ii).

From Lemma 5.3 (i) for $u \in F^1_{USCG}(X)$ and $\alpha \in (0, 1]$, $\lim_{\beta\to\alpha-} H([u]_\beta, [u]_\alpha) = 0$. So by Lemma 5.10 (ii), (ii-1) implies (ii-2). Clearly (ii-2) implies (ii-3). We shall complete the proof of (ii) by showing that (ii-3)⇒(ii-1). This follows from Lemma 5.9 (ii).

□

Theorem 5.14. *Let u be a fuzzy set in $F^1_{USCG}(X)$ and let u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. Then the following are equivalent:*

(i) $\lim_{n\to\infty} H_{\text{end}}(u_n, u) = 0$;

(ii) For each $\alpha \in (0, 1)$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ *and* $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$;

(iii) There is a dense subset P of $(0, 1)$ such that for each $\alpha \in P$, $\lim_{n\to\infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ *and* $\lim_{n\to\infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$;

(iv) For each $\alpha \in (0, 1) \setminus P_0(u)$, $\lim_{n\to\infty} H([u]_\alpha, [u_n]_\alpha) = 0$;

(v) There is a dense subset P of $(0, 1) \setminus P_0(u)$ such that for each $\alpha \in P$, $\lim_{n\to\infty} H([u]_\alpha, [u_n]_\alpha) = 0$;

(vi) There is a countable dense subset P of $(0, 1) \setminus P_0(u)$ such that for each $\alpha \in P$, $\lim_{n\to\infty} H([u]_\alpha, [u_n]_\alpha) = 0$;

(vii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *holds a.e. on* $\alpha \in (0, 1)$.

Proof. The desired results follow from Lemma 5.3 and Theorem 5.13. The proof is routine.

Note that $\lim_{n\to\infty} H_{\text{end}}(u_n, u) = 0$ if and only if $\lim_{n\to\infty} H^*(\text{end}\, u, \text{end}\, u_n) = 0$ and $\lim_{n\to\infty} H^*(\text{end}\, u_n, \text{end}\, u) = 0$. Hence Theorem 5.13 implies that (i)⇒(ii) and (iii)⇒(i). Clearly (ii)⇒(iii). So (i)⇔(ii)⇔(iii).

Since for each $\alpha \in (0,1) \setminus P_0(u)$, $\overline{\{u > \alpha\}} = [u]_\alpha$, then (ii)⇒(iv) is true. Clearly (iv)⇒(v)⇒(vi). Since $\overline{\{u > \alpha\}} \subseteq [u]_\alpha$, (vi)⇒(iii) is true.

From Lemma 5.3, we have $P_0(u)$ is at most countable, and therefore (iv)⇒(vii). Since $\overline{\{u > \alpha\}} \subseteq [u]_\alpha$, (vii)⇒(iii) is true. So the proof is completed.

□

Theorem 5.15. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. If there is a dense subset P of $(0,1)$ such that $H([u_n]_\alpha, [u]_\alpha) \to 0$ for each $\alpha \in P$, then $H_{\text{end}}(u_n, u) \to 0$.*

Proof. We proceed by contradiction. If $H_{\text{end}}(u_n, u) \not\to 0$, then there is an $\varepsilon > 0$ such that $H_{\text{end}}(u_{n_k}, u) > \varepsilon$ for a subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Suppose $H^*(\text{end}\, u_{n_k}, \text{end}\, u) > \varepsilon$. Then there exists a sequence $(x_{n_k}, \alpha_{n_k}) \in \text{end}\, u_{n_k}$ such that

$$d((x_{n_k}, \alpha_{n_k}), \text{end}\, u) > \varepsilon. \tag{10}$$

With no loss of generality we can assume $\alpha_{n_k} \to \alpha \geq \varepsilon$. Pick $\beta \in P$ satisfying $\alpha \in (\beta, \beta + \varepsilon/2)$. Then there exists K such that $\alpha_{n_k} \in (\beta, \beta + \varepsilon/2)$ for all $k \geq K$. Thus for each $k \geq K$,

$$\begin{aligned} & d((x_{n_k}, \alpha_{n_k}), \text{end}\, u) \\ & \quad \leq d((x_{n_k}, \beta), \text{end}\, u) + \varepsilon/2 \\ & \quad \leq H([u_{n_k}]_\beta, [u]_\beta) + \varepsilon/2 \end{aligned} \tag{11}$$

Note that $H([u_{n_k}]_\beta, [u]_\beta) \to 0$, thus (10) contradicts (11). So the supposition is false.

For $H^*(\text{end}\, u, \text{end}\, u_{n_k}) > \varepsilon$, we can similarly derive a contradiction.

□

Remark 5.16. It can be seen that Theorem 5.15 can also be deduced from Lemma 5.9 (iii). Fan (Lemma 1 in [4]) proved a result of Theorem 5.15 type.

Theorem 5.17. *Let u, u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. If $H_{\text{end}}(u_n, u) \to 0$, then $H([u_n]_\alpha, [u]_\alpha) \to 0$ for each $\alpha \in (0,1) \setminus P_0(u)$.*

Proof. Let $\alpha \in (0,1) \setminus P_0(u)$. Given $\varepsilon > 0$. Then there exists a $\delta(\alpha, \varepsilon) \in (0, \varepsilon/2)$ such that $[\alpha - \delta, \alpha + \delta] \subset [0,1]$ and

$$H([u]_\beta, [u]_\alpha) < \varepsilon/2 \tag{12}$$

for all $\beta \in [\alpha - \delta, \alpha + \delta]$.

From $H_{\text{end}}(u_n, u) \to 0$, there exists an $N(\delta)$ such that

$$H_{\text{end}}(u_n, u) < \delta \tag{13}$$

for all $n \geq N$. Thus

$$H^*([u_n]_\alpha, [u]_{\alpha-\delta}) < \delta < \varepsilon/2.$$

So, for each $n \geq N$,

$$\begin{aligned}
&H^*([u_n]_\alpha, [u]_\alpha) \\
&\leq H^*([u_n]_\alpha, [u]_{\alpha-\delta}) + H([u]_\alpha, [u]_{\alpha-\delta}) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned} \tag{14}$$

Similarly, it follows from (13) that

$$H^*([u]_{\alpha+\delta}, [u_n]_\alpha) < \delta < \varepsilon/2,$$

and then, for each $n \geq N$,

$$\begin{aligned}
&H^*([u]_\alpha, [u_n]_\alpha) \\
&\leq H([u]_\alpha, [u]_{\alpha+\delta}) + H^*([u]_{\alpha+\delta}, [u_n]_\alpha) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned} \tag{15}$$

Combined with (14) and (15),

$$H([u]_\alpha, [u_n]_\alpha) \to 0.$$

$\square$

Remark 5.18. Note that for each $\alpha \in (0,1) \setminus P_0(u)$, $\lim_{\lambda \to \alpha} H([u]_\alpha, [u]_\lambda) = 0$ and $[u]_\alpha = \overline{\{u > \alpha\}}$. Thus Theorem 5.17 can also be deduced from Lemma 5.10.

The following theorem gives a condition under which the H_{end} convergence of fuzzy sets can be decomposed to the Hausdorff metric convergence of certain α-cuts.

Theorem 5.19. *Let u be a fuzzy set in $F^1_{USCG}(X)$ and let u_n, $n = 1, 2, \ldots$, be fuzzy sets in $F^1_{USC}(X)$. Then the following are equivalent:*

(i) $H_{\text{end}}(u_n, u) \to 0$*;*

(ii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *holds a.e. on* $\alpha \in (0, 1)$*;*

(iii) $H([u_n]_\alpha, [u]_\alpha) \to 0$ *for all* $\alpha \in (0, 1) \setminus P_0(u)$*;*

(iv) There is a dense subset P of $(0, 1)\backslash P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \to 0$ for $\alpha \in P$;

(v) There is a countable dense subset P of $(0, 1)\backslash P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \to 0$ for $\alpha \in P$.

Proof. Theorem 5.17 implies that (i)⇒(iii). Clearly (iii)⇒(iv)⇒(v). Theorem 5.15 implies that (v)⇒(i) and (ii)⇒(i).

We shall complete the proof by showing that (iii)⇒(ii). This follows from the fact that $P_0(u)$ is at most countable, which is pointed out by Lemma 5.3. □

Remark 5.20. By Lemma 5.3, $P_0(u) = P(u)$ for $u \in F^1_{USCG}(X)$. So $P_0(u)$ can be replaced by $P(u)$ in Theorem 5.19.

Remark 5.21. Clearly, Theorem 5.14 implies Theorem 5.19.

$F^1_{USCGCON}(X)$ is a subset of $F^1_{USCG}(X)$ defined by

$F^1_{USCGCON}(X) := \{u \in F^1_{USCG}(X) :$ for each $\alpha \in (0,1]$, $[u]_\alpha$ is connected in $X\}$.

The statement A listed below is known (see Propositions 2.4 and 2.5 in [10]).

A Let C be a nonempty compact set in $\mathbb{R}^m$ and for $n = 1, 2, \ldots$, let C_n be a nonempty compact and connected set in $\mathbb{R}^m$. Then $H(C_n, C) \to 0$ if and only if $\lim^{(K)}_{n\to\infty} C_n = C$.

The following statement A^F is an immediate consequence of the above statement A and Theorems 4.14 and 5.19. Statement A^F is a special case of Theorem 9.2 in [10].

$\mathbf{A}^F$ Let u be a fuzzy set in $F^1_{USCG}(\mathbb{R}^m)$ and for $n = 1, 2, \ldots$, let u_n be a fuzzy set in $F^1_{USCGCON}(\mathbb{R}^m)$. Then $H_{\mathrm{end}}(u_n, u) \to 0$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.

The proof of statement A^F is routine, which is given as follows.

By let $X = \mathbb{R}^m$ in Theorems 5.19 and 4.14, we have
(i) $H_{\mathrm{end}}(u_n, u) \to 0$ if and only if $H([u_n]_\alpha, [u]_\alpha) \to 0$ holds a.e. on $\alpha \in (0, 1)$, and
(ii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ if and only if $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$ holds a.e. on $\alpha \in (0, 1)$.

For each $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, $[u]_\alpha \in K(\mathbb{R}^m)$, $[u_n]_\alpha \in K(\mathbb{R}^m)$ and $[u_n]_\alpha$ is connected in $\mathbb{R}^m$. Thus by statement A, for each $\alpha \in (0, 1]$, $H([u_n]_\alpha, [u]_\alpha) \to 0$ if and only if $[u]_\alpha = \lim^{(K)}_{n\to\infty}[u_n]_\alpha$. Combined this fact with the above clauses (i) and (ii), we have $H_{\mathrm{end}}(u_n, u) \to 0$ if and only if $\lim^{(\Gamma)}_{n\to\infty} u_n = u$.

However, the statement A^F would be false if $\mathbb{R}^m$ were replaced by a general metric space X.

Remark 5.22. Example Appendix A.4 shows that there exist u and u_n, $n = 1, 2, \ldots$ in $F^1_{USC}(\prod_{x\in(0,1]}[0, 3])$ such that
(i) $H_{\mathrm{send}}(u_n, u) \to 0$,
(ii) $(0, 1) \setminus P_0(u) = \emptyset$,
(iii) $H([u_n]_\alpha, [u]_\alpha) = 1$ for all $\alpha \in (0, 1]$ and $n = 1, 2, \ldots$,
(iv) $H_{\mathrm{end}}(u_n, u) \to 0$.

The above (i)-(iii) are shown in Example Appendix A.4. From Proposition 3.2, (i) implies (iv).

From (ii), for each $\{v_n\}$ in $F^1_{USC}(\prod_{x\in(0,1]}[0,3])$, the statement "$H([v_n]_\alpha, [u]_\alpha) \to 0$ for each $\alpha \in (0,1) \setminus P_0(u)$" is true. However "$H_{\rm end}(v_n, u) \to 0$" is not necessarily hold.

So from Example Appendix A.4 we know: the converse of the implication of Theorem 5.15 does not hold; the converse of the implication in Theorem 5.17 does not hold; "u be a fuzzy set in $F^1_{USCG}(X)$" can not be replaced by "u be a fuzzy set in $F^1_{USC}(X)$" in Theorem 5.19.

6. Relations among metrics on $F^1_{USC}(X)$

In this section, we discuss the relation among the $H_{\rm end}$ metric, the $H_{\rm send}$ metric, the d_∞ metric, and the d_p^* metric on $F^1_{USC}(X)$. Most of the results are proved based on the level characterizations of $H_{\rm end}$ given in Section 5.

For $u, v \in F^1_{USC}(X)$, the d_p distance given by

$$d_p(u,v) = \left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p}$$

is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$. In the sequel, we suppose that the d_p distance satisfying $p \geq 1$.

Since $H([u]_\alpha, [v]_\alpha)$ could be a non-measurable function of α on $[0,1]$ (see Example 2.13 in [12]), we introduce the d_p^* distance on $F^1_{USC}(X)$, $p \geq 1$, in [11], which is defined by

$$d_p^*(u,v) := \inf\{ \left(\int_0^1 f(\alpha)^p \, d\alpha \right)^{1/p} : f \text{ is a measurable function from } [0,1] \text{ to } \mathbb{R} \cup \{+\infty\}; \; f(\alpha) \geq H([u]_\alpha, [v]_\alpha) \text{ for } \alpha \in [0,1] \; \}$$

for $u, v \in F^1_{USC}(X)$.

d_p^* is an extended metric but probably not a metric on $F^1_{USC}(X)$. In this paper, we call d_p^* on $F^1_{USC}(X)$ the d_p^* metric for simplicity. See also Remark 3.3 in [11].

Clearly for $u, v \in F^1_{USC}(X)$,

$$d_\infty(u,v) \geq d_p^*(u,v). \tag{16}$$

The proof of (16) is routine. Set $d_\infty(u,v) = \xi \in \mathbb{R} \cup \{+\infty\}$. Define $f : [0,1] \to \mathbb{R} \cup \{+\infty\}$ by $f(\alpha) = \xi$ for each $\alpha \in [0,1]$. Hence f is a measurable

function from $[0,1]$ to $\mathbb{R} \cup \{+\infty\}$ and $f(\alpha) \geq H([u]_\alpha, [v]_\alpha)$ for $\alpha \in [0,1]$. So $d_p^*(u,v) \leq \left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} = \xi$. Thus (16) is true.

If $d_p(u,v)$ is well-defined for $u,v \in F_{USC}^1(X)$, then $d_p^*(u,v) = d_p(u,v)$. The d_p^* metric is an expansion of the d_p distance on $F_{USC}^1(X)$.

The d_p distance is well-defined on $F_{USC}^1(\mathbb{R}^m)$ (see [11], Proposition 2.2 in [12]). The d_p distance is well-defined on $F_{USCG}^1(X)$ (see [11], Proposition 2.7 in [12]). We can see that the d_p distance is a metric on $F_{USCB}^1(X)$.

Let $u \in F_{USCG}^1(\mathbb{R})$ defined by

$$u(x) = \begin{cases} 0, & x < 1, \\ 1, & x = 1, \\ 1/n, & x \in (n^2, (n+1)^2], \ n = 1, 2, \ldots. \end{cases}$$

Then $d_p(u, \widehat{1}_{F(\mathbb{R})}) = +\infty$. The d_p distance on $F_{USCG}^1(\mathbb{R}^m)$ is an extended metric but not a metric, and the d_p distance on $F_{USC}^1(\mathbb{R}^m)$ is an extended metric but not a metric. In this paper, we call the d_p distance on $F_{USC}^1(\mathbb{R}^m)$ or $F_{USCG}^1(X)$ the d_p metric for simplicity.

Theorem 6.1. *Let (X,d) be a metric space and let $u,v \in F_{USC}^1(X)$. Then*

$$d_p^*(u,v) \geq \left(\frac{H_{\text{end}}(u,v)^{p+1}}{p+1}\right)^{1/p}. \tag{17}$$

Proof. To show the desired result, we only need to show that for each $r > 0$, if $H_{\text{end}}(u,v) > r$ then $d_p^*(u,v) \geq \left(\frac{r^{p+1}}{p+1}\right)^{1/p}$.

Let $r > 0$. Assume that $H_{\text{end}}(u,v) > r$. Then without loss of generality we suppose that $H^*(\text{end}\, u, \text{end}\, v) > r$, then there is an $(x,\beta) \in \text{end}\, u$, such that $d((x,\beta), \text{end}\, v) > r$. This implies that $\beta > r$ and $d(x, [v]_\alpha) > r - (\beta - \alpha)$ when $\alpha \in [\beta - r, \beta]$. Hence $H^*([u]_\alpha, [v]_\alpha) > r - (\beta - \alpha)$ when $\alpha \in [\beta - r, \beta]$.

Let f be a measurable function on $[0,1]$ with $f(\alpha) \geq H([u]_\alpha, [v]_\alpha)$ for $\alpha \in [0,1]$. Then

$$\begin{aligned} \left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} &\geq \left(\int_{\beta-r}^{\beta} f(\alpha)^p \, d\alpha\right)^{1/p} \\ &> \left(\int_{\beta-r}^{\beta} (r - (\beta - \alpha))^p \, d\alpha\right)^{1/p} \end{aligned}$$

$$= \left(\frac{r^{p+1}}{p+1}\right)^{1/p}.$$

So $d_p^*(u,v) \geq \left(\frac{r^{p+1}}{p+1}\right)^{1/p}$.

□

The "=" can be obtained in (17).

Example 6.2. Define u and v in $F^1_{USCB}(\mathbb{R})$ as

$$u(x) = \begin{cases} 1, & x = 0, \\ 0.5 - x, & x \in (0, 0.5], \\ 0, & \text{otherwise}, \end{cases} \quad v(x) = \begin{cases} 1, & x = 0, \\ 0.5, & x \in (0, 0.5], \\ 0, & \text{otherwise}. \end{cases}$$

Then $H_{\text{end}}(u, v) = 0.5$ and

$$H([u]_\alpha, [v]_\alpha) = \begin{cases} 0, & \alpha \in (0.5, 1], \\ \alpha, & \alpha \in [0, 0.5]. \end{cases}$$

Thus $d_p(u,v) = (\int_0^{0.5} \alpha^p \, d\alpha)^{1/p} = \left(\frac{0.5^{p+1}}{p+1}\right)^{1/p} = \left(\frac{H_{\text{end}}(u,v)^{p+1}}{p+1}\right)^{1/p}$.

We can see that the following Theorem 6.3 can also be deduced by the Theorem 6.1.

Theorem 6.3. *Let $u \in F^1_{USC}(X)$ and for each positive integer n, let $u_n \in F^1_{USC}(X)$. If $d_p^*(u_n, u) \to 0$, then $H_{\text{end}}(u_n, u) \to 0$.*

Proof. We prove by contradiction. If $H_{\text{end}}(u_n, u) \not\to 0$, then there is an $\varepsilon > 0$ and a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$H_{\text{end}}(v_n, u) \geq \varepsilon. \tag{18}$$

From the definition of d_p^*, there exist a sequence $\{f_n\}$ of measurable function from $[0,1]$ to $\mathbb{R} \cup \{+\infty\}$ such that for each $n \in \mathbb{N}$

$$H([v_n]_\alpha, [u]_\alpha) \leq f_n(\alpha) \text{ for all } \alpha \in [0,1], \tag{19}$$

$$\left(\int_0^1 f_n(\alpha)^p d\,\alpha\right)^{1/p} \leq \frac{n+1}{n} d_p^*(v_n, u). \tag{20}$$

Since $d_p^*(v_n, u) \to 0$, by (20), we have $\left(\int_0^1 f_n(\alpha)^p d\,\alpha\right)^{1/p} \to 0$. Thus there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ a.e. converges to 0 on $[0, 1]$. Hence by (19), $H([v_{n_k}]_\alpha, [u]_\alpha) \to 0$ holds a.e. on $\alpha \in (0, 1)$. From Theorem 5.15, this implies $H_{\text{end}}(v_{n_k}, u) \to 0$, which contradicts (18).

□

Theorem 4.1 in [12] says that for $u \in F^1_{USCG}(X)$ and $v \in F^1_{USC}(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$. So $d_p^*(u, v) = d_p(u, v)$ for $u \in F^1_{USCG}(X)$ and $v \in F^1_{USC}(X)$.

Theorem 6.4. *Suppose that $u \in F^1_{USCG}(X)$ and $u_n \in F^1_{USC}(X)$, $n = 1, 2, \ldots$, and that there is a measurable function F on $[0, 1]$ such that $\int_0^1 F^p(\alpha)\, d\alpha < +\infty$ and $H([u_n]_\alpha, [u]_\alpha) \le F(\alpha)$ for $n = 1, 2, \ldots$. If $H_{\text{end}}(u_n, u) \to 0$, then $d_p(u_n, u) \to 0$.*

Proof. By Theorem 5.19, $H_{\text{end}}(u_n, u) \to 0$ if and only if $H([u_n]_\alpha, [u]_\alpha) \to 0$ holds a.e. on $\alpha \in (0, 1)$. So the desired result follows from the Lebesgue's Dominated Convergence Theorem.

□

Corollary 6.5. *Let $u \in F^1_{USCG}(X)$ and for $n = 1, 2, \ldots$, let $u_n \in F^1_{USC}(X)$. If $H_{\text{end}}(u_n, u) \to 0$ and $\bigcup_{n=1}^{+\infty} [u_n]_0$ is bounded, then $d_p(u_n, u) \to 0$.*

Proof. Since $H_{\text{end}}(u_n, u) \to 0$ and $\bigcup_{n=1}^{+\infty} [u_n]_0$ is bounded, then from Remark 4.6, $[u]_0 \subseteq \liminf_{n\to\infty} [u_n]_0 \subseteq \overline{\bigcup_{n=1}^{+\infty} [u_n]_0}$ is bounded. Hence there is an $M \in \mathbb{R}$ such that $d_\infty(u_n, u) \le M$; that is, for all $\alpha \in [0, 1]$, $H([u_n]_\alpha, [u]_\alpha) \le M$. Thus by Theorem 6.4, $d_p(u_n, u) \to 0$.

□

Proposition 6.6. *Let $u \in F^1_{USCB}(X)$ and for each positive integer n, let $u_n \in F^1_{USC}(X)$. Then $H_{\text{send}}(u_n, u) \to 0$ if and only if $H([u_n]_0, [u]_0) \to 0$ and $d_p(u_n, u) \to 0$.*

Proof. From Proposition 3.2, $H_{\text{send}}(u_n, u) \to 0$ if and only if $H([u_n]_0, [u]_0) \to 0$ and $H_{\text{end}}(u_n, u) \to 0$. To prove the desired result, we only need to show that

$$H([u_n]_0, [u]_0) \to 0 \text{ and } d_p(u_n, u) \to 0 \Leftrightarrow H([u_n]_0, [u]_0) \to 0 \text{ and } H_{\text{end}}(u_n, u) \to 0.$$

From Theorem 6.3, "$\Rightarrow$" is true. To show "$\Leftarrow$", suppose that $H([u_n]_0, [u]_0) \to 0$ and $H_{\rm end}(u_n, u) \to 0$. Then there exist an $N \in \mathbb{N}$ such that $\bigcup_{n\geq N}[u_n]_0$ is bounded. Thus by Corollary 6.5, $d_p(u_n, u) \to 0$. So"$\Leftarrow$" is true.

$\square$

The above results and the examples in this paper show that the following statements are true for u, u_n in $F^1_{USC}(X)$, $n = 1, 2, \ldots$.

(i) $d_p^*(u_n, u) \to 0$ imply $H_{\rm end}(u_n, u) \to 0$ (a stronger conclusion is given in Theorem 6.1).

(ii) Suppose that $u \in F^1_{USCB}(X)$. By Proposition 6.6, $H_{\rm send}(u_n, u) \to 0$ if and only if $H([u_n]_0, [u]_0]) \to 0$ and $d_p(u_n, u) \to 0$.

Suppose that u and $\{u_n\}$ is in $F^1_{USCB}(X)$. Example 6.7 below shows that $d_p(u_n, u) \to 0$ does not necessarily imply that $\bigcup_{n=1}^{+\infty}[u_n]_0$ is bounded. Hence $d_p(u_n, u) \to 0$ does not necessarily imply $H_{\rm send}(u_n, u) \to 0$ since $H_{\rm send}(u_n, u) \to 0$ implies that $\bigcup_{n=1}^{+\infty}[u_n]_0$ is bounded for u and $\{u_n\}$ in $F^1_{USCB}(X)$ (In fact, by Theorem 7.15, $H_{\rm send}(u_n, u) \to 0$ implies that $\bigcup_{n=1}^{+\infty}[u_n]_0$ is relatively compact). Take u, $\{u_n\}$ in Remark 4.6, we can see that $d_p(u_n, u) \to 0$ and $\bigcup_{n=1}^{+\infty}[u_n]_0$ is compact need not imply $H_{\rm send}(u_n, u) \to 0$.

(iii) Suppose that $u \in F^1_{USCG}(X)$. By Theorem 6.4, under certain conditions, $H_{\rm end}(u_n, u) \to 0$ imply $d_p(u_n, u) \to 0$. However, even the sequence $\{u_n\}$ is in $F^1_{USCG}(X)$, $H_{\rm send}(u_n, u) \to 0$ does not necessarily imply $d_p(u_n, u) \to 0$. See the following Example 6.8. By Corollary 6.5, $H_{\rm end}(u_n, u) \to 0$ and $\bigcup_{n=1}^{+\infty}[u_n]_0$ is bounded imply $d_p(u_n, u) \to 0$.

(iv) Suppose that $u \in F^1_{USC}(X)$. Even $H_{\rm send}(u_n, u) \to 0$ and $\bigcup_{n=1}^{+\infty}[u_n]_0$ is bounded do not necessarily imply $d_p^*(u_n, u) \to 0$. Below we illustrate that a such example is given in Example Appendix A.4.

Example Appendix A.4 shows that there exist u and u_n, $n = 1, 2, \ldots$ in $F^1_{USC}(\prod_{x\in(0,1]}[0, 3])$ such that
(i) $H_{\rm send}(u_n, u) \to 0$,
(ii) $[u]_0 = [u_n]_0 = \{\xi \in \prod_{x\in(0,1]}[0, 1] : \xi_x \to 0 \text{ as } x \to 0\}$ for all $n = 1, 2, \ldots$,
(iii) $H([u_n]_\alpha, [u]_\alpha) = 1$ for all $\alpha \in (0, 1]$ and $n = 1, 2, \ldots$,
(iv) $\sup\{d(x, y) : x, y \in \bigcup_{n=1}^{+\infty}[u_n]_0\} = 1$, and so $\bigcup_{n=1}^{+\infty}[u_n]_0$ is bounded,
(v) $d_p(u_n, u) = 1$ for all $n = 1, 2, \ldots$, and so $d_p(u_n, u) \not\to 0$.

The above (i)-(iii) are shown in Example Appendix A.4. (ii) implies (iv). (iii) implies (v).

Example 6.7. Let $u = \widehat{0}_{F(\mathbb{R})} \in F^1_{USCB}(\mathbb{R})$. For $n = 1, 2, \ldots$, define $u_n \in F^1_{USCB}(\mathbb{R})$ as

$$u_n = \begin{cases} 1, & x = 0, \\ 1/n^2, & 0 < x \le n^{1/p}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $d_p(u_n, u) = \left(\int_0^{1/n^2} (n^{1/p})^p \, d\alpha\right)^{1/p} = (\frac{1}{n})^{1/p} \to 0$. However, $\bigcup_{n=1}^{+\infty} [u_n]_0 = [0, +\infty)$ is not bounded.

Example 6.8. Define $u \in F^1_{USCG}(\mathbb{R})$ as follows:

$$[u]_\alpha = [0, 1/\alpha].$$

For each $n = 1, 2, \ldots$, define $u_n \in F^1_{USCG}(\mathbb{R})$ as follows:

$$[u_n]_\alpha = \begin{cases} [0, 1/\alpha], & \alpha \in (1/n, 1], \\ [0, 1/\alpha + n^2], & \alpha \in [0, 1/n]. \end{cases}$$

Then $H_{\text{send}}(u_n, u) \to 0$ and $d_p(u_n, u) = n^{2-1/p} \not\to 0$.

In [11], we obtain that the Skorokhod metric convergence imply the sendograph metric convergence on $F^1_{USC}(X)$ (see Theorem 8.1 in [11]), and that Skorokhod metric convergence need not imply the d_p convergence on a subset of $F^1_{USCG}(X)$ (see the end of Section 5 in [11]). From the above conclusions, we can also deduce that the sendograph metric convergence need not imply the d_p convergence on $F^1_{USCG}(X)$.

7. Characterizations of compactness in $(F^1_{USCG}(X), H_{\text{end}})$ and $(F^1_{USCB}(X), H_{\text{send}})$

Based on the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X), H_{\text{end}})$ and $(F^1_{USCB}(X), H_{\text{send}})$.

- A subset Y of a topological space Z is said to be *compact* if for every set I and every family of open sets, O_i, $i \in I$, such that $Y \subset \bigcup_{i \in I} O_i$ there exists a finite family $O_{i_1}, O_{i_2} \ldots, O_{i_n}$ such that $Y \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_n}$. In the case of a metric topology, the criterion for compactness becomes that any sequence in Y has a subsequence convergent in Y.

- A *relatively compact* subset Y of a topological space Z is a subset with compact closure. In the case of a metric topology, the criterion for relative compactness becomes that any sequence in Y has a subsequence convergent in X.

- Let (X, d) be a metric space. A set U in X is *totally bounded* if and only if, for each $\varepsilon > 0$, it contains a finite ε approximation, where an ε approximation to U is a subset S of U such that $d(x, S) < \varepsilon$ for each $x \in U$. An ε approximation to U is also called an ε-net of U.

Let (X, d) be a metric space. A set U is compact in (X, d) implies that U is relatively compact in (X, d), which in turn implies that U is totally bounded in (X, d).

We use $(\widetilde{X}, \widetilde{d})$ to denote the completion of (X, d). We see (X, d) as a subspace of $(\widetilde{X}, \widetilde{d})$. Let $S \subseteq \widetilde{X}$. The symbol $\overline{\widetilde{S}}$ is used to denote the closure of S in $(\widetilde{X}, \widetilde{d})$.

As defined in Section 2, we have $K(\widetilde{X})$, $C(\widetilde{X})$, $F^1_{USC}(\widetilde{X})$, $F^1_{USCG}(\widetilde{X})$, etc. according to $(\widetilde{X}, \widetilde{d})$. For example,

$$F^1_{USC}(\widetilde{X}) := \{u \in F(\widetilde{X}) : [u]_\alpha \in C(\widetilde{X}) \text{ for all } \alpha \in [0, 1]\},$$
$$F^1_{USCG}(\widetilde{X}) := \{u \in F(\widetilde{X}) : [u]_\alpha \in K(\widetilde{X}) \text{ for all } \alpha \in (0, 1]\}.$$

If there is no confusion, we also use H to denote the Hausdorff metric on $C(\widetilde{X})$ induced by $\widetilde{d}$. We also use H to denote the Hausdorff metric on $C(\widetilde{X} \times [0, 1])$ induced by $\overline{\widetilde{d}}$. We also use $H_{\rm end}$ to denote the endograph metric on $F^1_{USC}(\widetilde{X})$ given by using H on $C(\widetilde{X} \times [0, 1])$.

Clearly, the induced metric on $F^1_{USCG}(X)$ by the $H_{\rm end}$ on $F^1_{USC}(X)$ is the same as the induced metric on $F^1_{USCG}(X)$ by the $H_{\rm end}$ on $F^1_{USC}(\widetilde{X})$.

We see $(F^1_{USCG}(X), H_{\rm end})$ as a metric subspace of $(F^1_{USCG}(\widetilde{X}), H_{\rm end})$.

7.1. Characterizations of compactness in $(K(X), H)$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(K(X), H)$. The results in this subsection are basis for contents in the sequel.

Theorem 7.1. *Let* (X, d) *be complete and let* $\{C_n\}$ *be a Cauchy sequence in* $(K(X), H)$*. Put* $D_n = \bigcup_{l=1}^{n} C_l$ *and* $D = \overline{\bigcup_{l=1}^{+\infty} C_l}$*. Then* $D \in K(X)$ *and* $H(D_n, D) \to 0$*.*

Proof. Note that for $k > j$,

$$H(D_k, D_j) \leq \max\{H(C_i, C_j) : i = j+1, \ldots, k\}.$$

So $\{D_n\}$ is a Cauchy sequence in $(K(X), H)$. From Theorem 2.2, $(K(X), H)$ is complete, and thus by Theorem 3.3, $\{D_n\}$ converges to $\limsup_{n\to\infty} D_n = D \in K(X)$.

$\square$

Theorem 7.2. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then $\mathcal{D}$ is totally bounded in $(K(X), H)$ if and only if $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is totally bounded in (X, d).*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$.

Necessity. To show that $\mathbf{D}$ is totally bounded. We only need to show that each sequence in $\mathbf{D}$ has a Cauchy subsequence.

Given a sequence $\{x_n\}$ in $\mathbf{D}$. Suppose that $x_n \in C_n \in \mathcal{D}$. Since $\mathcal{D}$ is totally bounded, then $\{C_n\}$ has a Cauchy subsequence $\{C_{n_k}\}$. Hence, by Theorem 7.1, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\widetilde{X})$. Thus $\{x_n\}$ has a Cauchy subsequence.

Sufficiency. If $\mathbf{D}$ is totally bounded in X, then $\widetilde{\overline{\mathbf{D}}}$ is in $K(\widetilde{X})$. So, by Theorem 2.2, $(K(\widetilde{\overline{\mathbf{D}}}), H)$ is compact, and thus $\mathcal{D}$ is totally bounded.

$\square$

Theorem 7.3. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then $\mathcal{D}$ is relatively compact in $(K(X), H)$ if and only if $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d).*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$.

Necessity. To show that $\mathbf{D}$ is relatively compact. We only need to show that each sequence in $\mathbf{D}$ has a convergent subsequence in X.

Given a sequence $\{x_n\}$ in $\mathbf{D}$. Suppose that $x_n \in C_n \in \mathcal{D}$. Since $\mathcal{D}$ is relatively compact, then $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ which converges to C in $K(X)$. Hence, by Theorem 7.1, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\widetilde{X})$ (In fact, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(X)$). So $\{x_{n_k}\}$ has a subsequence which converges to x in $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$, and thus $x \in C \subset X$.

Sufficiency. If $\mathbf{D}$ is relatively compact in X, then $\overline{\mathbf{D}}$ is in $K(X)$, and therefore $(K(\overline{\mathbf{D}}), H)$ is compact. Thus $\mathcal{D} \subset K(\overline{\mathbf{D}})$ is relatively compact in $(K(X), H)$.

□

Lemma 7.4. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. If $\mathcal{D}$ is compact in $(K(X), H)$, then $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d).*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$. To show that $\mathbf{D}$ is compact. We only need to show that each sequence in $\mathbf{D}$ has a subsequence which converges to a point in $\mathbf{D}$.

Given a sequence $\{x_n\}$ in $\mathbf{D}$. Suppose that $x_n \in C_n \in \mathcal{D}$. Since $\mathcal{D}$ is compact, then $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ converges to $C \in \mathcal{D}$. Hence, by Theorem 7.1, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\widetilde{X})$ (In fact, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\mathbf{D})$). So $\{x_{n_k}\}$ has a subsequence which converges to x in $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$. Thus $x \in C \subset \mathbf{D}$.

□

Remark 7.5. The converse of the implication in Lemma 7.4 does not hold. Let $(X, d) = \mathbb{R}$ and $\mathcal{D} = \{[0, x] : x \in (0.3, 1]\} \subset K(\mathbb{R})$. Then $\mathbf{D} = [0, 1] \in K(\mathbb{R})$. But $\mathcal{D}$ is not compact in $(K(\mathbb{R}), H)$.

Theorem 7.6. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then the following are equivalent:*
(i) $\mathcal{D}$ is compact in $(K(X), H)$;
(ii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) and $\mathcal{D}$ is closed in $(K(X), H)$;
(iii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d) and $\mathcal{D}$ is closed in $(K(X), H)$.

Proof. Note that $\mathcal{D}$ is compact in $(K(X), H)$ if and only if $\mathcal{D}$ is relatively compact and closed in $(K(X), H)$. Then from Theorem 7.3 we have (i)⇔(ii). Clearly (iii)⇒(ii). We shall complete the proof by showing that (i)⇒(iii), which can be deduced by Lemma 7.4.

□

Remark 7.7. After we gave conclusions and their proofs in this section, we found that Theorem 7.3 is Proposition 5 in [6]. Since we can't find the proof of Proposition 5, we give our proof here.

7.2. Characterizations of compactness in $(F^1_{USCG}(X), H_{\rm end})$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(F^1_{USCG}(X), H_{\rm end})$.

Let U be a subset of (X, d). We say a subset S of X is a *weak ε-net* of U if $d(x, S) < \varepsilon$ for each $x \in U$.

We can see that an ε-net of U must be a weak ε-net of U. An ε-net of U is included in U, however a weak ε-net of U is not necessarily be included in U. For convenience, we use the term weak ε-net of a set U in (X, d).

For $x \in X$ and $\varepsilon > 0$, let $B(x, \varepsilon) := \{z \in X : d(x, z) < \varepsilon\}$.

The following known conclusions on totally bounded are useful in this paper.

- A set U in X is totally bounded if and only if for each sequence $\{x_n\}$ in U, it has a subsequence $\{x_{n_k}\}$ which is a Cauchy sequence.

- A set U in X is totally bounded if and only if for each $\varepsilon > 0$, there is a finite weak ε-net of U.

Below we give a proof of the last conclusion mentioned above, although we are convinced that the proof for this conclusion was already been given. Readers who think it unnecessary to give the proof or this conclusion is obvious can skip this proof.

Clearly the necessity is true since an ε-net of U is a weak ε-net of U. To show the sufficiency, it suffices to show how to construct a finite ε-net of U via a finite weak $\varepsilon/2$-net of U.

Let $S_{\varepsilon/2}$ be a finite weak $\varepsilon/2$-net of U. Set $S'_{\varepsilon/2} := \{y \in S_{\varepsilon/2} : B(y, \varepsilon/2) \cap U \neq \emptyset\}$. Clearly $S'_{\varepsilon/2}$ is a finite weak $\varepsilon/2$-net of U. Then for each $y \in S'_{\varepsilon/2}$, we choose an x_y in $B(y, \varepsilon/2) \cap U$. Let $T_\varepsilon := \{x_y : y \in S'_{\varepsilon/2}\}$. We claim that T_ε is a finite ε-net of U. Clearly T_ε is a finite subset of U. To complete the proof, we only need to show that $d(z, T_\varepsilon) < \varepsilon$ for each $z \in U$. Let $z \in U$. Then there is a $y \in S'_{\varepsilon/2}$ with $d(z, y) < \varepsilon/2$, and thus $d(z, T_\varepsilon) \leq d(z, x_y) \leq d(z, y) + d(y, x_y) < \varepsilon$.

Suppose that U is a subset of $F^1_{USC}(X)$ and $\alpha \in [0, 1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u \in U} [u]_\alpha$, and

- $U_\alpha := \{[u]_\alpha : u \in U\}$.

Theorem 7.8. *Let U be a subset of $F^1_{USCG}(X)$. Then U is totally bounded in $(F^1_{USCG}(X), H_{\rm end})$ if and only if $U(\alpha)$ is totally bounded in (X,d) for each $\alpha \in (0,1]$.*

Proof. ***Necessity***. Suppose that U is totally bounded in $(F^1_{USCG}(X), H_{\rm end})$. Let $\alpha \in (0,1]$. To show that $U(\alpha)$ is totally bounded in X, we only need to show that each sequence in $U(\alpha)$ has a Cauchy subsequence.

Given a sequence $\{x_n\} \subset U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \ldots$. Then $\{u_n\}$ has a Cauchy subsequence $\{u_{n_l}\}$. So given $\varepsilon \in (0, \alpha)$, there is a $K(\varepsilon) \in \mathbb{N}$ such that

$$H_{\rm end}(u_{n_l}, u_{n_K}) < \varepsilon$$

for all $l \geq K$. Thus

$$H^*([u_{n_l}]_\alpha, [u_{n_K}]_{\alpha-\varepsilon}) < \varepsilon \tag{21}$$

for all $l \geq K$. From (21) and the arbitrariness of ε, $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$ is totally bounded in (X,d). Thus $\{x_{n_l}\}$, which is a subsequence of $\{x_n\}$, has a Cauchy subsequence, and so does $\{x_n\}$.

In the following, we give a more detailed proof for the above conclusion that $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$ is totally bounded in (X,d), although we think the proof given above for this conclusion is sufficient.

To show the desired result, it suffices to show that for each $\lambda > 0$, there exists a finite weak λ-net of $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$.

Let $\lambda > 0$. Set $\varepsilon = \min\{\lambda/2, \alpha/2\}$. Then $\varepsilon \in (0, \alpha)$. Hence there is a $K(\varepsilon)$ such that (21) holds for all $l \geq K$. Since $\bigcup_{l=1}^{K} [u_{n_l}]_{\alpha-\varepsilon}$ is compact, there is a finite ε-net $\{z_j\}_{j=1}^m$ of $\bigcup_{l=1}^{K} [u_{n_l}]_{\alpha-\varepsilon}$. We claim that $\{z_j\}_{j=1}^m$ is a finite weak λ-net of $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$.

Let $z \in \bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$. If $z \in \bigcup_{l=1}^{K} [u_{n_l}]_\alpha$, then clearly $d(z, \{z_j\}_{j=1}^m) < \varepsilon < \lambda$. If $z \in \bigcup_{l=K+1}^{+\infty} [u_{n_l}]_\alpha$, then by (21), there exists a $y_z \in [u_{n_K}]_{\alpha-\varepsilon}$ such that $d(z, y_z) < \varepsilon$, and so $d(z, \{z_j\}_{j=1}^m) \leq d(z, y_z) + d(y_z, \{z_j\}_{j=1}^m) < 2\varepsilon \leq \lambda$.

Sufficiency. Suppose that $U(\alpha)$ is totally bounded in X for each $\alpha \in (0,1]$. By Theorem 7.2, $U(\alpha)$ is totally bounded in X if and only if U_α is totally bounded in $(K(X), H)$. Thus, by Theorem 2.2, we have the following affirmation:

- Given $\alpha \in (0,1]$. For each sequence $\{[u_n]_\alpha, n = 1, 2, \ldots\}$ in U_α, it has a subsequence $\{[u_{n_k}]_\alpha, k = 1, 2, \ldots\}$ which converges to $u_\alpha \in K(\widetilde{X})$ with respect to the Hausdorff metric H.

To prove that U is totally bounded, it suffices to show that each sequence in U has a convergent subsequence in $(F^1_{USCG}(\widetilde{X}), H_{\rm end})$. Suppose that $\{u_n\}$ is a sequence in U. Based on the above affirmation and Theorem 5.19, and proceeding similarly to the proof of the "Sufficiency part" of Theorem 7.1 in [10], it can be shown that $\{u_n\}$ has a subsequence $\{v_n\}$ which converges to $v \in F^1_{USCG}(\widetilde{X})$ with respect to $H_{\rm end}$.

A sketch of the proof of the existence of $\{v_n\}$ and v is given as follows.

First, we construct a subsequence $\{v_n\}$ of $\{u_n\}$ such that $[v_n]_q$ converges to $u_q \in K(\widetilde{X})$ according to the Hausdorff metric H for all $q \in \mathbb{Q}'$, where $\mathbb{Q}' = \mathbb{Q} \cap (0, 1]$. Then we show that $v \in F^1_{USCG}(\widetilde{X})$ with $[v]_\alpha = \bigcap_{q<\alpha, q\in\mathbb{Q}'} u_q$ for all $\alpha \in (0, 1]$ satisfies that $H_{\rm end}(v_n, v) \to 0$.

$\square$

Remark 7.9. Some of the implications in the proofs of this paper are actually the equivalent. For example, in the proof of Theorem 7.8, $U(\alpha)$ is totally bounded in X for each $\alpha \in (0, 1]$ is equivalent to the affirmation after the "•"

Theorem 7.10. *Let U be a subset of $F^1_{USCG}(X)$. Then U is relatively compact in $(F^1_{USCG}(X), H_{\rm end})$ if and only if $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$.*

Proof. ***Necessity***. Suppose that U is relatively compact. Given $\alpha \in (0, 1]$. To show that $U(\alpha)$ is relatively compact in X, we only need to show that each sequence in $U(\alpha)$ has a convergent subsequence in X.

Let $\{x_n\}$ be a sequence in $U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \ldots$. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in F^1_{USCG}(X)$ such that $H_{\rm end}(u_{n_k}, u) \to 0$. So, by Theorem 5.19, $H([u_{n_k}]_\alpha, [u]_\alpha) \xrightarrow{\text{a.e.}} 0$, and therefore there is a $\beta \in (0, \alpha)$ such that $H([u_{n_k}]_\beta, [u]_\beta) \to 0$. Hence by Theorem 7.3, $\bigcup_{k=1}^{+\infty} [u_{n_k}]_\beta$ is relatively compact in X. Thus $\{x_{n_k}\}$ has a convergent subsequence in X, and so does $\{x_n\}$.

Sufficiency. Suppose that $U(\alpha)$ is relatively compact in X for each $\alpha \in (0, 1]$. To show that U is relatively compact in $(F^1_{USCG}(X), H_{\rm end})$, we only need to show that each sequence in U has a convergent subsequence in $(F^1_{USCG}(X), H_{\rm end})$.

By Theorem 7.3, $U(\alpha)$ is relatively compact in X if and only if U_α is relatively compact in $K(X)$. Thus, we have the following affirmation:

- Given $\alpha \in (0,1]$. For each sequence $\{[u_n]_\alpha, n = 1,2,\ldots\}$ in U_α, it has a subsequence $\{[u_{n_k}]_\alpha, k = 1,2,\ldots\}$ which converges to $u_\alpha \in K(X)$ with respect to the Hausdorff metric H.

The remaining proof is similar to the corresponding part of the "Sufficiency part" of Theorem 7.8.

We can also prove that U is relatively compact in $(F^1_{USCG}(X), H_{\rm end})$ as follows. From the "Sufficiency part" of Theorem 7.8, we know that for each sequence $\{u_n\}$ in U, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ which converges to $v \in F^1_{USCG}(\widetilde{X})$. From Theorem 5.19 and the above statement after the "•", we thus know that $v \in F^1_{USCG}(X)$.

□

Theorem 7.11. *Let U be a subset of $F^1_{USCG}(X)$. Then the following are equivalent:*

(i) U is compact in $(F^1_{USCG}(X), H_{\rm end})$;

(ii) $U(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$ and U is closed in $(F^1_{USCG}(X), H_{\rm end})$;

(iii) $U(\alpha)$ is compact in (X,d) for each $\alpha \in (0,1]$ and U is closed in $(F^1_{USCG}(X), H_{\rm end})$.

Proof. By Theorem 7.10, (i) ⇔ (ii). Obviously (iii) ⇒ (ii). We shall complete the proof by showing that (i) ⇒ (iii). To do this, suppose that U is compact. To verify (iii), from the equivalence of (i) and (ii), we only need to show that $U(\alpha)$ is closed in (X,d) for each $\alpha \in (0,1]$.

Let $\alpha \in (0,1]$ and let $\{x_n\}$ be a sequence in $U(\alpha)$ with $x_n \to x$. Suppose that $x_n \in [u_n]_\alpha$ and $u_n \in U$ for $n = 1,2,\ldots$. Then there exist subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in U$ such that $H_{\rm end}(u_{n_k}, u) \to 0$. So by Remark 3.4 $\lim^{(\Gamma)}_{n\to\infty} u_{n_k} = u$ and therefore by Theorem 4.5, $\limsup_{n\to\infty}[u_{n_k}]_\alpha \subseteq [u]_\alpha$. Hence $x \in [u]_\alpha$, and thus $x \in U(\alpha)$.

We can also show $x \in [u]_\alpha \subseteq U(\alpha)$ in the following way. From Theorem 5.14, $H([u_{n_k}]_\alpha, [u]_\alpha) \to 0$ holds for $\alpha \in (0,1) \setminus P_0(u)$. If $\alpha \in (0,1) \setminus P_0(u)$, then $x \in [u]_\alpha$. If $\alpha \in \{1\} \cup P_0(u)$, then for all $\beta \in (0,\alpha) \setminus P_0(u)$, $x \in [u]_\beta$. Thus $x \in [u]_\alpha$.

□

7.3. Characterizations of compactness in $(P^1_{USCB}(X), H_{\rm send})$ and $(F^1_{USCB}(X), H_{\rm send})$

In this subsection, we give the characterizations of totally bounded sets, relatively compact sets and compact sets in $(P^1_{USCB}(X), H_{\rm send})$. Then, by treating $(F^1_{USCB}(X), H_{\rm send})$ as a metric subspace of $(P^1_{USCB}(X), H_{\rm send})$, we give the characterizations of totally bounded sets and compact sets in $(F^1_{USCB}(X), H_{\rm send})$. The characterization of relatively compact sets in $(F^1_{USCB}(X), H_{\rm send})$ has already been given in [6].

Suppose that U is a subset of $P^1_{USC}(X)$ and $\alpha \in [0,1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u\in U} \langle u\rangle_\alpha$, and
- $U_\alpha := \{\langle u\rangle_\alpha : u \in U\}$.

Theorem 7.12. *Suppose that U is a subset of $P^1_{USCB}(X)$. Then U is totally bounded in $(P^1_{USCB}(X), H_{\rm send})$ if and only if $U(0)$ is totally bounded in (X, d).*

Proof. ***Necessity***. Suppose that U is totally bounded. By clause (ii) of Theorem 3.1, U_0 is totally bounded in $(K(X), H)$. From Theorem 7.2, this is equivalent to $U(0)$ is totally bounded in (X, d).

Sufficiency. Suppose that $U(0)$ is totally bounded. To show that U is totally bounded in $(P^1_{USCB}(X), H_{\rm send})$, we only need to prove that each sequence in U has a Cauchy subsequence with respect to $H_{\rm send}$.

Let $\{u_n\}$ be a sequence in U. Note that $U(\alpha)$ is totally bounded for each $\alpha \in [0,1]$. Then by Theorem 7.8, $\{\overleftarrow{u_n}\}$ has a Cauchy subsequence $\{v_n\}$ in $(F^1_{USCB}(X), H_{\rm end})$. From Theorem 7.2, $\{v_n\}$ has a subsequence $\{w_n\}$ such that $\{[w_n]_0\}$ is a Cauchy sequence in $(K(X), H)$. Thus by clauses (iii) of Theorem 3.1, $\{w_n\}$ is a Cauchy sequence in $(P^1_{USCB}(X), H_{\rm send})$.

□

Theorem 7.13. *Suppose that U is a subset of $F^1_{USCB}(X)$. Then U is totally bounded in $(F^1_{USCB}(X), H_{\rm send})$ if and only if $U(0)$ is totally bounded in (X, d).*

Proof. Note that U is totally bounded in $(F^1_{USCB}(X), H_{\rm send})$ if and only if $\overrightarrow{U}$ is totally bounded in $(P^1_{USCB}(X), H_{\rm send})$, and that $U(0) = \overrightarrow{U}(0)$. So the desired result follows from Theorem 7.12.

□

Theorem 7.14. *Suppose that U is a subset of $P^1_{USCB}(X)$. Then U is relatively compact in $(P^1_{USCB}(X), H_{\rm send})$ if and only if $U(0)$ is relatively compact in X.*

Proof. ***Necessity***. Suppose that U is relatively compact. Then by clauses (ii) of Theorem 3.1, U_0 is relatively compact in $(K(X), H)$. By Theorem 7.3, $U(0)$ is relatively compact in X.

Sufficiency. To prove that U is relatively compact, it suffices to show that each sequence in U has a convergent subsequence in $(P^1_{USCB}(X), H_{\rm send})$.

Let $\{u_n\}$ be a sequence in U. Since $U(0)$ is relatively compact in X, then $U(\alpha)$ is relatively compact in X for each $\alpha \in [0,1]$. By Theorems 7.10 and 7.3, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, an $u \in F^1_{USCG}(X)$ and a $u_0 \in K(X)$ such that $H_{\rm end}(\overleftarrow{u_{n_k}}, u) \to 0$ and $H(\langle u_{n_k}\rangle_0, u_0) \to 0$.

Set $w \in P^1_{USCB}(X)$ given by

$$\langle w\rangle_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (0,1], \\ u_0, & \alpha = 0. \end{cases}$$

Then $u = \overleftarrow{w}$, $H_{\rm end}(u_{n_k}, w) = H_{\rm end}(\overleftarrow{u_{n_k}}, u) \to 0$ and $H(\langle u_{n_k}\rangle_0, \langle w\rangle_0) = H(\langle u_{n_k}\rangle_0, u_0) \to 0$ for $n = 1, 2, \ldots$ Thus from (iii) or (iv) of Theorem 3.1, $\{u_{n_k}\}$ converges to w in $(P^1_{USCB}(X), H_{\rm send})$.

$\square$

- $u \in F^1_{USC}(X)$ is said to be right-continuous at 0 if $\lim_{\delta\to 0+} H([u]_\delta, [u]_0) = 0$.
- $U \subset F^1_{USC}(X)$ is said to be equi-right-continuous at 0 if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $H([u]_\delta, [u]_0) < \varepsilon$ for all $u \in U$.

By Lemma 5.2, for each $u \in F^1_{USCB}(X)$, u is right-continuous at 0.

Theorem 7.15 below is presented in [6].

Theorem 7.15. *[6] Suppose that U is a subset of $F^1_{USCB}(X)$. Then U is relatively compact in $(F^1_{USCB}(X), H_{\rm send})$ if and only if $U(0)$ is relatively compact in X and U is equi-right-continuous at 0.*

$\overrightarrow{F^1_{USCB}(X)}$ need not be a closed set of $P^1_{USCB}(X)$. For instance, $F^1_{USCB}(D)$ given in Example 8.4 is not a closed set of $P^1_{USCB}(D)$. We can see that $\overrightarrow{F^1_{USCB}(X)}$ is a closed set of $P^1_{USCB}(X)$ if and only if X has only one element.

For a set U in $F^1_{USCB}(X)$, suppose that

(a) U is relatively compact in $(F^1_{USCB}(X), H_{\rm send})$;

(b) $\overrightarrow{U}$ is relatively compact in $(P^1_{USCB}(X), H_{\rm send})$;

(c) The topological closure of $\overrightarrow{U}$ in $(P^1_{USCB}(X), H_{\rm send})$ is a subset of $\overrightarrow{F^1_{USCB}(X)}$.

Then (a) holds if and only if (b) and (c) hold.

$\overrightarrow{F^1_{USCB}(X)}$ is closed in $P^1_{USCB}(X)$ if and only if for each set U in $F^1_{USCB}(X)$, (c) holds.

The following Proposition 7.16 illustrates the role of the condition "U is equi-right-continuous at 0" in the characterization of the relative compactness for a set U in $(F^1_{USCB}(X), H_{\rm send})$ given in Theorem 7.15.

For $w \in P^1_{USCB}(X)$, the following are equivalent: (i) $w \in \overrightarrow{F^1_{USCB}(X)}$; (ii) $w \in \overrightarrow{F^1_{USC}(X)}$; (iii) $\lim_{\delta\to 0+} H(\langle w\rangle_\delta, \langle w\rangle_0)$=0.

Proposition 7.16. *Let U be a subset of $F^1_{USCB}(X)$. Suppose the following conditions (i), (ii) and (iii):*
(i) U is relatively compact in $(F^1_{USCB}(X), H_{\rm send})$;
(ii) U is equi-right-continuous at 0;
(iii) The topological closure of $\overrightarrow{U}$ in $(P^1_{USCB}(X), H_{\rm send})$ is a subset of $\overrightarrow{F^1_{USCB}(X)}$.

Then the condition (i) implies the condition (ii), and the condition (ii) implies the condition (iii). If $\overrightarrow{U}$ is relatively compact in $(P^1_{USCB}(X), H_{\rm send})$, then the conditions (i), (ii) and (iii) are equivalent to each other.

Proof. By Theorem 7.15, we know that (i)$\Rightarrow$(ii).

To show (ii)$\Rightarrow$(iii). Suppose that $\{\overrightarrow{u_n}\}$ converges to u in $(P^1_{USCB}(X), H_{\rm send})$. Then $\lim_{n\to\infty} H(\langle\overrightarrow{u_n}\rangle_0, \langle u\rangle_0) = 0$; that is, $\lim_{n\to\infty} H([u_n]_0, \langle u\rangle_0) = 0$.

Note that $\{u_n\}$ converges to $\overleftarrow{u}$ in $(F^1_{USCB}(X), H_{\rm end})$. By Theorem 5.19, $H([u_n]_\alpha, [\overleftarrow{u}]_\alpha) \to 0$ holds a.e. on $\alpha \in (0, 1)$. From U is equi-right-continuous at 0, we have $\lim_{n\to\infty} H([u_n]_0, [\overleftarrow{u}]_0) = 0$.

Thus $\langle u\rangle_0 = [\overleftarrow{u}]_0$, and hence $u \in \overrightarrow{F^1_{USCB}(X)}$. So (ii)$\Rightarrow$(iii).

If $\overrightarrow{U}$ is relatively compact in $(P^1_{USCB}(X), H_{\rm send})$, then clearly (iii)$\Rightarrow$(i), and thus the conditions (i), (ii) and (iii) are equivalent to each other.

□

Remark 7.17. For conditions (i), (ii) and (iii) in Proposition 7.16, (ii) does not imply (i); (iii) does not imply (ii).

Let $\{u_n\}$ be a sequence of fuzzy sets in $F^1_{USCB}(\mathbb{R})$ defined by $u_n(x) = [0, n]_{\mathbb{R}}$, $n = 1, 2, \ldots$. Then $\{u_n\}$ is equi-right-continuous at 0 but $\{u_n\}$ is not relatively compact in $(F^1_{USCB}(X), H_{\rm send})$. So (ii) does not imply (i).

Let $\{v_n\}$ be a sequence of fuzzy sets in $F^1_{USCB}(\mathbb{R})$ defined by

$$v_n(x) = \begin{cases} 1, & x \in [0, n], \\ 1/n, & x \in [-n, 0], \\ 0, & x \in \mathbb{R} \setminus [-n, n], \end{cases} \qquad n = 1, 2, \ldots.$$

Then $\{\overrightarrow{v_n}\}$ has no limit in $(P^1_{USCB}(X), H_{\text{send}})$ and hence is closed in $(P^1_{USCB}(X), H_{\text{send}})$. However $\{v_n\}$ is not equi-right-continuous at 0. So (iii) does not imply (ii).

Remark 7.18. Theorem 7.14 and Proposition 7.16 imply Theorem 7.15. This is because by Proposition 7.16 we can obtain that for a subset U of $F^1_{USCB}(X)$, U is relatively compact in $(F^1_{USCB}(X), H_{\text{send}})$ if and only if $\overrightarrow{U}$ is relatively compact in $(P^1_{USCB}(X), H_{\text{send}})$ and U is equi-right-continuous at 0.

Theorem 7.19. *Suppose that U is a subset of $P^1_{USCB}(X)$. Then the following are equivalent:*
(i) U is compact in $(P^1_{USCB}(X), H_{\text{send}})$;
(ii) U is closed in $(P^1_{USCB}(X), H_{\text{send}})$, and $U(0)$ is relatively compact in (X, d);
(iii) U is closed in $(P^1_{USCB}(X), H_{\text{send}})$, and $U(0)$ is compact in (X, d).

Proof. By Theorem 7.14, we obtain that (i)⇔(ii). Clearly (iii)⇒(ii). We shall complete the proof by showing that (i)⇒(iii). To do this, suppose that U is compact in $(P^1_{USCB}(X), H_{\text{send}})$. Then U is closed in $(P^1_{USCB}(X), H_{\text{send}})$. To verify (iii), we only need to show that $U(0)$ is compact in (X, d).

By clause (ii) of Theorem 3.1, U_0 is compact in $(K(X), H)$. Thus by Theorem 7.6, $U(0)$ is compact in (X, d).

□

Theorem 7.20. *Suppose that U is a subset of $F^1_{USCB}(X)$. Then the following are equivalent:*
(i) U is compact in $(F^1_{USCB}(X), H_{\text{send}})$;
(ii) U is closed in $(F^1_{USCB}(X), H_{\text{send}})$, $U(0)$ is relatively compact in X and U is equi-right-continuous at 0;
(iii) U is closed in $(F^1_{USCB}(X), H_{\text{send}})$, $U(0)$ is compact in X and U is equi-right-continuous at 0.

Proof. By Theorem 7.15, we obtain that (i)⇔(ii). Clearly (iii)⇒(ii). We shall complete the proof by showing that (i)⇒(iii). To do this, suppose that U is compact in $(F^1_{USCB}(X), H_{\text{send}})$. Since (i) implies (ii), to verify (iii) we only need to show that $U(0)$ is compact in X.

Note that U is compact in $(F^1_{USCB}(X), H_{\text{send}})$ if and only if $\overrightarrow{U}$ is compact in $(P^1_{USCB}(X), H_{\text{send}})$. Thus by Theorems 7.19, $U(0) = \overrightarrow{U}(0)$ is compact in X.

□

Remark 7.21. For a subset U of $F^1_{USCB}(X)$,
(a) U satisfies (i) of Theorem 7.20 if and only if $\overrightarrow{U}$ satisfies (i) of Theorem 7.19;
(b) U satisfies (ii) of Theorem 7.20 if and only if $\overrightarrow{U}$ satisfies (ii) of Theorem 7.19;
(c) U satisfies (iii) of Theorem 7.20 if and only if $\overrightarrow{U}$ satisfies (iii) of Theorem 7.19.

Clauses (b) and (c) can be obtained by using Proposition 7.16 and Theorem 7.14. Theorem 7.19 and clauses (a), (b) and (c) imply Theorem 7.20.

Since U is compact in $(F^1_{USCB}(X), H_{\rm send})$ if and only if $\overrightarrow{U}$ is compact in $(P^1_{USCB}(X), H_{\rm send})$, we can use Theorem 7.19 to judge the compactness of a set U in $(F^1_{USCB}(X), H_{\rm send})$.

8. Completions of $(F^1_{USCB}(X), H_{\rm send})$ and $(F^1_{USCG}(X), H_{\rm end})$

In this section, we show that $(P^1_{USCB}(\widetilde{X}), H_{\rm send})$ is a completion of $(F^1_{USCB}(X), H_{\rm send})$. We also show that $(F^1_{USCG}(\widetilde{X}), H_{\rm end})$ is a completion of $(F^1_{USCB}(X), H_{\rm end})$, and thus a completion of $(F^1_{USCG}(X), H_{\rm end})$.

Theorem 8.1. *Let (X, d) be a metric space. Then the following are equivalent:*

(i) (X, d) is complete;

(ii) $(F^1_{USCG}(X), H_{\rm end})$ is complete.

Proof. (i) $\Rightarrow$ (ii). Let $\{u_n\}$ be a Cauchy sequence of $(F^1_{USCG}(X), H_{\rm end})$. Then $U = \{u_n, n = 1, 2, \ldots\}$ is total bounded in $(F^1_{USCG}(X), H_{\rm end})$. So from the proof the sufficiency part of Theorem 7.8, we know that $\{u_n\}$ has a convergent subsequence in $(F^1_{USCG}(X), H_{\rm end})$, and thus $\{u_n\}$ is convergent in $(F^1_{USCG}(X), H_{\rm end})$.

(ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a Cauchy sequence in X. Note that $H_{\rm end}(\widehat{x}, \widehat{y}) = \min\{d(x, y), 1\}$ for $x, y \in X$. Then $\{\widehat{x_n}\}$ is a Cauchy sequence in $(F^1_{USCG}(X), H_{\rm end})$, and therefore $\{\widehat{x_n}\}$ converges to $u \in F^1_{USCG}(X)$. Thus there exists an $x \in X$ such that $[u]_\alpha = \{x\}$ for all $\alpha \in [0, 1]$ (i.e. $u = \widehat{x}$) and $d(x_n, x) \to 0$.

□

Remark 8.2. (ii) $\Rightarrow$ (i) in Theorem 8.1 can also be shown as follows

(X, d) is complete if and only if (X, d^*) is complete, where $d^*(x, y) = \min\{d(x, y), 1\}$ for $x, y \in X$. Note that $H_{\text{end}}(\widehat{x}, \widehat{y}) = d^*(x, y)$. So the desired result follows from the fact that (X, d^*) is isometric to the closed subspace $(\widehat{X}, H_{\text{end}})$ of $(F^1_{USCG}(X), H_{\text{end}})$, where $\widehat{X} := \{\widehat{x} : x \in X\}$.

Proposition 8.3. *Let (X, d) be a metric space. Then the following are equivalent:*
(i) (X, d) is complete;
(ii) $(F^1_{USCB}(X), d_\infty)$ is complete.

Proof. (X, d) is isometric to $(\widehat{X}, d_\infty)$, which is a closed subspace of $(F^1_{USCB}(X), d_\infty)$. So (ii)⇒(i) is proved.

To show (i)⇒(ii), suppose that (X, d) is complete. Let $\{u_n\}$ be a Cauchy sequence in $(F^1_{USCB}(X), d_\infty)$. Then for each $\alpha \in [0, 1]$, $\{[u_n]_\alpha\}$ is a Cauchy sequence in $(K(X), H)$, and hence there is an $u(\alpha) \in K(X)$ such that $H([u_n]_\alpha, u(\alpha)) \to 0$.

As $\{u_n\}$ is a Cauchy sequence in $(F^1_{USCB}(X), d_\infty)$, $H([u_n]_\alpha, u(\alpha))$ converges uniformly to 0 on $\alpha \in [0, 1]$, denoted by

$$H([u_n]_\alpha, u(\alpha)) \rightrightarrows 0\,([0, 1]). \tag{22}$$

Clearly $u(\alpha) \subseteq u(\beta)$ for $0 \le \beta \le \alpha \le 1$.

If there is a $u \in F^1_{USCB}(X)$ such that $[u]_\alpha = u(\alpha)$, then by (22), $d_\infty(u_n, u) \to 0$, and so the proof is complete. To prove the existence of a such $u \in F^1_{USCB}(X)$, we only need to show that $\{u(\alpha), \alpha \in [0, 1]\}$ has the following properties:
(i) for each $\alpha \in (0, 1]$, $u(\alpha) = \bigcap_{\beta<\alpha} u(\beta)$, and
(ii) $u(0) = \overline{\bigcup_{\gamma>0} u(\gamma)}$.

Since for $n = 1, 2, \ldots$ and $\alpha \in (0, 1]$, $\lim_{\beta\to\alpha-} H([u_n]_\alpha, [u_n]_\beta) = 0$ and $\lim_{\gamma\to 0+} H([u_n]_\gamma, [u_n]_0) = 0$, then by (22), $\lim_{\beta\to\alpha-} H(u(\alpha), u(\beta)) = 0$ and $\lim_{\gamma\to 0+} H(u(\gamma), u(0)) = 0$. Thus by Lemma 5.2, for each $\alpha \in (0, 1]$, $u(\alpha) = \bigcap_{\beta<\alpha} u(\beta)$, and $u(0) = \overline{\bigcup_{\gamma>0} u(\gamma)}$. So (i) and (ii) are true.

□

Even if (X, d) is complete, $(F^1_{USCB}(X), H_{\text{send}})$ need not be complete. See Example 8.4 below.

Example 8.4. let $D = \{0, 1\}$ be a metric subspace of $\mathbb{R}$. Then D is complete. Let $u_n \in F^1_{USCB}(D)$, $n = 1, 2, \ldots$, be defined as

$$u_n(x) = \begin{cases} 1, & x = 0, \\ 1/n, & x = 1. \end{cases}$$

Then $\{\overrightarrow{u_n}\}$ converges to $u \in P^1_{USCB}(D)$ defined by

$$\langle u \rangle_\alpha = \begin{cases} \{0\}, & \alpha \in (0, 1], \\ \{0, 1\}, & \alpha = 0. \end{cases}$$

Thus $\{u_n\}$ is a Cauchy sequence in $(F^1_{USCB}(D), H_{\mathrm{send}})$ and has no limit in $(F^1_{USCB}(D), H_{\mathrm{send}})$. So $(F^1_{USCB}(D), H_{\mathrm{send}})$ is not complete.

We can see that $(F^1_{USCB}(X), H_{\mathrm{send}})$ is complete if and only if X has only one element.

Theorem 8.5 below discusses the completeness of $(P^1_{USCB}(X), H_{\mathrm{send}})$ and then Theorem 8.6 below gives the completion of $(F^1_{USCB}(X), H_{\mathrm{send}})$.

Theorem 8.5. *Let (X, d) be a metric space. Then the following are equivalent:*

(i) X is complete;

(ii) $(P^1_{USCB}(X), H_{\mathrm{send}})$ is complete.

Proof. (i) $\Rightarrow$ (ii). Let $\{u_n\}$ be a Cauchy sequence in $(P^1_{USCB}(X), H_{\mathrm{send}})$. Then $\{\overleftarrow{u_n}\} \subseteq F^1_{USCB}(X)$, and by (i) of Theorem 3.1, $H_{\mathrm{end}}(\overleftarrow{u_n}, \overleftarrow{u_m}) = H_{\mathrm{end}}(u_n, u_m) \leq H_{\mathrm{send}}(u_n, u_m)$ for $n, m = 1, 2, \ldots$. Hence $\{\overleftarrow{u_n}\}$ is a Cauchy sequence in $(F^1_{USCG}(X), H_{\mathrm{end}})$. From Theorem 8.1, there is an $u \in F^1_{USCG}(X)$ such that $\{\overleftarrow{u_n}\}$ converges to u in $(F^1_{USCG}(X), H_{\mathrm{end}})$.

By (ii) of Theorem 3.1, $H(\langle u_n \rangle_0, \langle u_m \rangle_0) \leq H_{\mathrm{send}}(u_n, u_m)$ for $n, m = 1, 2, \ldots$. So $\{\langle u_n \rangle_0\}$ is a Cauchy sequence in $(K(X), H)$. From Theorem 2.2, there is an $u_0 \in K(X)$ such that $\{\langle u_n \rangle_0\}$ converges to u_0 in $(K(X), H)$.

Set $w \in P^1_{USCB}(X)$ given by

$$\langle w \rangle_\alpha = \begin{cases} [u]_\alpha, & \alpha > 0, \\ u_0, & \alpha = 0. \end{cases}$$

Then $u = \overleftarrow{w}$, $H_{\mathrm{end}}(u_n, w) = H_{\mathrm{end}}(\overleftarrow{u_n}, u)$ and $H(\langle u_n \rangle_0, \langle w \rangle_0) = H(\langle u_n \rangle_0, u_0)$ for $n = 1, 2, \ldots$ Thus from (iii) or (iv) of Theorem 3.1, $\{u_n\}$ converges to w in $(P^1_{USCB}(X), H_{\mathrm{send}})$.

(ii) $\Rightarrow$ (i). Note that $d(x,y) = H_{\rm send}(\widehat{x}, \widehat{y})$. So the desired result follows from the fact that (X,d) is isometric to a closed subspace of $(P^1_{USCB}(X), H_{\rm send})$. $\square$

Theorem 8.6. $(P^1_{USCB}(\widetilde{X}), H_{\rm send})$ *is a completion of* $(F^1_{USCB}(X), H_{\rm send})$.

Proof. From Theorem 8.5, $(P^1_{USCB}(\widetilde{X}), H_{\rm send})$ is complete. To show that $(P^1_{USCB}(\widetilde{X}), H_{\rm send})$ is a completion of $(F^1_{USCB}(X), H_{\rm send})$, we only need to show that for each $u \in P^1_{USCB}(\widetilde{X})$ and each $\varepsilon > 0$, there is a $w \in F^1_{USCB}(X)$ such that $H_{\rm send}(u, \overrightarrow{w}) \leq \varepsilon$. To show this is equivalent to show the following affirmations (a) and (b):

(a) For each $u \in P^1_{USCB}(\widetilde{X})$ and each $\varepsilon > 0$, there exists a $v \in F^1_{USCB}(\widetilde{X})$ such that $H_{\rm send}(u, \overrightarrow{v}) \leq \varepsilon$.

(b) For each $v \in F^1_{USCB}(\widetilde{X})$ and each $\varepsilon > 0$, there exists a $w \in F^1_{USCB}(X)$ such that $d_\infty(v,w) \leq \varepsilon$. Then by (1) and (4), $H_{\rm send}(v,w) \leq \varepsilon$ and $H_{\rm end}(v,w) \leq \varepsilon$.

Let $u \in P^1_{USCB}(\widetilde{X})$. Define $u_\varepsilon \in F^1_{USCB}(\widetilde{X})$, $\varepsilon > 0$, given by

$$[u_\varepsilon]_\alpha = \begin{cases} \langle u \rangle_\alpha, & \alpha \in (\varepsilon, 1], \\ \langle u \rangle_0, & \alpha \in [0, \varepsilon]. \end{cases}$$

Then $H_{\rm send}(u, \overrightarrow{u_\varepsilon}) \leq \varepsilon$. So affirmation (a) is proved.

Let $v \in F^1_{USCB}(\widetilde{X})$. We can choose a finite subset C_0 of X such that $H(C_0, [v]_0) < \varepsilon$. Define

$$C_\alpha := \{x \in C_0 : d(x, [v]_\alpha) \leq \varepsilon\}, \ \alpha \in (0,1]. \tag{23}$$

We affirm that $\{C_\alpha : \alpha \in [0,1]\}$ has the following properties
(i) $C_\alpha \neq \emptyset$ for all $\alpha \in [0,1]$.
(ii) $H(C_\alpha, [v]_\alpha) \leq \varepsilon$ for all $\alpha \in [0,1]$.
(iii) $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$ for all $\alpha \in (0,1]$.
(iv) $C_0 = \bigcup_{\alpha>0} C_\alpha = \overline{\bigcup_{\alpha>0} C_\alpha}$.

For each $y \in [v]_\alpha$, there exists $z_y \in C_0$ such that $d(y, z_y) = d(y, C_0) < \varepsilon$. Hence $z_y \in C_\alpha$ and thus $C_\alpha \neq \emptyset$. So (i) is true.

To show (ii), we only need to show that $H(C_\alpha, [v]_\alpha) \leq \varepsilon$ for $0 < \alpha \leq 1$. Let $\alpha \in (0,1]$. From (23), $H^*(C_\alpha, [v]_\alpha) \leq \varepsilon$. In the following, we show that $H^*([v]_\alpha, C_\alpha) < \varepsilon$. In fact, for each $y \in [v]_\alpha$, $d(y, C_\alpha) \leq d(y, z_y) = d(y, C_0)$

(hence $d(y, C_\alpha) = d(y, C_0)$). Thus $H^*([v]_\alpha, C_\alpha) \le H([v]_0, C_0) < \varepsilon$. So (ii) is proved.

Let $\alpha \in (0,1]$. Clearly $C_\alpha \subseteq \bigcap_{\beta<\alpha} C_\beta$. By Lemma 5.2, $\lim_{\beta\to\alpha-} H([v]_\alpha, [v]_\beta) = 0$. So for each $x \in X$, $d(x, [v]_\alpha) = \lim_{\beta\to\alpha-} d(x, [v]_\beta)$, and hence $C_\alpha \supseteq \bigcap_{\beta<\alpha} C_\beta$. Thus $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$. So (iii) is true.

Let $x \in C_0$. Then $d(x, [v]_0) < \varepsilon$. Since $[v]_0 = \overline{\cup_{\alpha>0}[v]_\alpha}$, there exists $\alpha > 0$ such that $d(x, [v]_\alpha) < \varepsilon$ (in fact, for each $x \in X$, $d(x, [v]_0) = \inf_{\alpha>0} d(x, [v]_\alpha)$), and thus $x \in C_\alpha$. So (iv) is proved.

Set $w \in F(X)$ given by $[w]_\alpha = C_\alpha$ for all $\alpha \in [0,1]$. Then by (i), (iii) and (iv), $w \in F^1_{USCB}(X)$. From (ii), we have $d_\infty(v, w) \le \varepsilon$, and then $H_{\rm send}(v, w) \le \varepsilon$. So affirmation (b) is proved.

□

Proposition 8.7. *$(F^1_{USCB}(\widetilde{X}), d_\infty)$ is a completion of $(F^1_{USCB}(X), d_\infty)$.*

Proof. By Proposition 8.3, $(F^1_{USCB}(\widetilde{X}), d_\infty)$ is complete. So from affirmation (b) in the proof of Theorem 8.6, we have the desired result.

□

Theorem 8.8. *$(P^1_{USCB}(\widetilde{X}), H_{\rm send})$ is a completion of $(P^1_{USCB}(X), H_{\rm send})$.*

Proof. $(F^1_{USCB}(X), H_{\rm send})$ can be seen as a metric subspace of $(P^1_{USCB}(X), H_{\rm send})$. $(P^1_{USCB}(X), H_{\rm send})$ is a metric subspace of $(P^1_{USCB}(\widetilde{X}), H_{\rm send})$. So the desired result follows from Theorem 8.6.

□

Theorem 8.9. *$(F^1_{USCG}(\widetilde{X}), H_{\rm end})$ is a completion of $(F^1_{USCB}(X), H_{\rm end})$.*

Proof. From Theorem 8.1, $(F^1_{USCG}(\widetilde{X}), H_{\rm end})$ is complete. To prove the desired result, it suffices to show that $F^1_{USCB}(X)$ is dense in $(F^1_{USCG}(\widetilde{X}), H_{\rm end})$. By affirmation (b) in the proof of Theorem 8.6, to verify this it is enough to show that for each $u \in F^1_{USCG}(\widetilde{X})$ and each $\varepsilon > 0$, there is a $v \in F^1_{USCB}(\widetilde{X})$ such that $H_{\rm end}(u, v) \le \varepsilon$.

Let $u \in F^1_{USCG}(\widetilde{X})$. Define $u^\varepsilon \in F^1_{USCB}(\widetilde{X})$, $\varepsilon > 0$, given by

$$[u^\varepsilon]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (\varepsilon, 1], \\ [u]_\varepsilon, & \alpha \in [0, \varepsilon]. \end{cases}$$

Then $H_{\rm end}(u, u^\varepsilon) \le \varepsilon$.

□

Corollary 8.10. *$(F^1_{USCG}(\widetilde{X}), H_{\rm end})$ is a completion of $(F^1_{USCG}(X), H_{\rm end})$.*

Proof. Since $F^1_{USCB}(X) \subseteq F^1_{USCG}(X) \subseteq F^1_{USCG}(\widetilde{X})$, the desired result follows from Theorem 8.9.

□

9. Conclusions

In this paper, we discuss the properties and relations of $H_{\rm end}$ metric and $H_{\rm send}$ metric on fuzzy sets in a metric space X.

To aid discussion, we introduce the sets $P^1_{USC}(X)$ and $P^1_{USCB}(X)$. $P^1_{USCB}(X)$ is a subset of $P^1_{USC}(X)$. The $F^1_{USC}(X)$ and $F^1_{USCB}(X)$ can be viewed as the subsets of $P^1_{USC}(X)$ and $P^1_{USCB}(X)$, respectively. We define the $H_{\rm send}$ distance and the $H_{\rm end}$ distance on $P^1_{USC}(X)$, and give the relations among the $H_{\rm send}$ distance, the $H_{\rm end}$ distance and the Kuratowski convergence on $P^1_{USC}(X)$. Then, as corollaries, we obtain the relations among the $H_{\rm send}$ metric, the $H_{\rm end}$ metric and the Γ-convergence on $F^1_{USC}(X)$.

We give the level characterizations of $H_{\rm end}$ convergence and Γ-convergence on $F^1_{USC}(X)$. By using the above results including the level characterizations of the $H_{\rm end}$ convergence, we give the relations among the $H_{\rm end}$ metric, the $H_{\rm send}$ metric and the d^*_p metric. We point out that the values of the metrics can be directly compared among certain ones in the supremum metric d_∞, the $H_{\rm end}$ metric, the $H_{\rm send}$ metric and the d^*_p metric.

Based on above results, we give characterizations of compactness and completions of two kinds of fuzzy set spaces $(F^1_{USCG}(X), H_{\rm end})$ and $(F^1_{USCB}(X), H_{\rm send})$, respectively. We also investigate characterizations of compactness and completions of $(P^1_{USCB}(X), H_{\rm send})$. $(F^1_{USCB}(X), H_{\rm send})$ can be treated as a metric subspace of $(P^1_{USCB}(X), H_{\rm send})$.

Note that $\mathbb{R}^m$ is complete and for a set V in $\mathbb{R}^m$, the following are equivalent: (i) V is bounded; (ii) V is totally bounded; (iii) V is relatively compact. We can obtain the characterizations of compactness and completions of $(F^1_{USCB}(\mathbb{R}^m), H_{\rm send})$, $(F^1_{USCG}(\mathbb{R}^m), H_{\rm end})$ and $(P^1_{USCB}(\mathbb{R}^m), H_{\rm send})$ by using that of $(F^1_{USCB}(X), H_{\rm send})$, $(F^1_{USCG}(X), H_{\rm end})$ and $(P^1_{USCB}(X), H_{\rm send})$ given in this paper.

The results in this paper have potential applications in fuzzy set research involving $H_{\rm end}$ metric and $H_{\rm send}$ metric.

Appendix A. Counterexamples

In this section, we give an example to illustrate the conclusions in Sections 4, 5 and 6.

Let (X_j, d_j), $j \in J$, be metric spaces. Define an extended metric d on $\prod_{j\in J} X_j$ as
$$d(x,y) := \sup\{d_j(x_j, y_j) : j \in J\} \tag{A.1}$$
for $x = (x_j)_{j\in J}$ and $y = (y_j)_{j\in J}$.

We use the symbol $\prod_{j\in J}(X_j, d_j)$ to denote the extended metric space $(\prod_{j\in J} X_j, d)$. If not mentioned specially, we suppose by default that $\prod_{j\in J} X_j$ is endowed with the extended metric d given by (A.1).

Let $u_j \in F(X_j)$, $j \in J$. Define $u \in F(\prod_{j\in J} X_j)$ as

$$[u]_\alpha = \prod_{j\in J} [u_j]_\alpha \text{ for each } \alpha \in (0,1]. \tag{A.2}$$

We use $\boldsymbol{\prod_{j\in J} u_j}$ to denote the fuzzy set u in $\prod_{j\in J} X_j$ given by (A.2).

In [12], we show that $u = \prod_{j\in J} u_j$ is well-defined and give the conclusion stated in Theorem Appendix A.2. For an extended metric space (Y, ρ), we define

$$F_{USC}(Y) = \{u \in F(Y) : [u]_\alpha \text{ is closed in } (Y,\rho) \text{ for } \alpha \in (0,1]\}$$
$$F^1_{USC}(Y) = \{u \in F(Y) : [u]_\alpha \text{ is nonempty and closed in } (Y,\rho) \text{ for } \alpha \in (0,1]\}$$

Theorem Appendix A.1. *[12] Let J be a set, and for each $j \in J$, let (X_j, d_j) be a metric space. If $u_j \in F_{USC}(X_j)$ for each $j \in J$, then $u = \prod_{j\in J} u_j$ is a fuzzy set in $F_{USC}(\prod_{j\in J} X_j)$.*

Theorem Appendix A.2. *Let J be a set, and for each $j \in J$, let (X_j, d_j) be a metric space. If $u_j \in F^1_{USC}(X_j)$ for each $j \in J$, then $u = \prod_{j\in J} u_j$ is a fuzzy set in $F^1_{USC}(\prod_{j\in J} X_j)$.*

Proof. The desired result follows immediately from Theorem Appendix A.1 and the definition of $\prod_{j\in J} u_j$.

□

Theorem Appendix A.3. *Let J be a set, and for each $j \in J$, let (X_j, d_j) be a metric space, $u_j \in F(X_j)$ and $u = \prod_{j\in J} u_j$. Then for each $\xi = (\xi_j)_{j\in J} \in \prod_{j\in J} X_j$, $u(\xi) = \inf_{j\in J} u_j(\xi_j)$.*

Proof. For each $\alpha \in (0,1]$, by (A.2)

$$\begin{aligned}
&u(\xi) \geq \alpha \\
&\Leftrightarrow \xi \in [u]_\alpha = \prod_{j\in J} [u_j]_\alpha \\
&\Leftrightarrow \text{for each } j \in J,\ \xi_j \in [u_j]_\alpha \\
&\Leftrightarrow \text{for each } j \in J,\ u_j(\xi_j) \geq \alpha \\
&\Leftrightarrow \inf_{j\in J} u_j(\xi_j) \geq \alpha.
\end{aligned}$$

Thus we obtain the desired result.

□

$[0,3]$ can be seen as a metric subspace of $\mathbb{R}$. So $\prod_{x\in(0,1]}[0,3]$ is a metric subspace with the metric d given by (A.1).

The following Example Appendix A.4 shows that there exist u and u_n, $n = 1, 2, \ldots$ in $F^1_{USC}(\prod_{x\in(0,1]}[0,3])$ such that $H_{\mathrm{send}}(u_n, u) \to 0$, $\{[u_n]_\alpha\}$ does not Kuratowski converge to $[u]_\alpha$ when $\alpha \in (0,1]$, and $H([u_n]_\alpha, [u]_\alpha) = 1$ for all $\alpha \in (0,1]$ and $n = 1, 2, \ldots$.

Example Appendix A.4. For $x \in (0,1]$, define $u_x \in F^1_{USC}([0,3])$ as follows

$$u_x(t) = \begin{cases} 1, & t = 0, \\ x, & t \in (0,1], \\ 0, & t \notin [0,1]. \end{cases}$$

Put

$$u = \prod_{x\in(0,1]} u_x.$$

Then, by Theorem Appendix A.2, $u \in F^1_{USC}(\prod_{x\in(0,1]}[0,3])$ and

$$[u]_\alpha = \begin{cases} \prod_{x\in(0,1)}\{0\}, & \alpha = 1, \\ \prod_{x\in(0,\alpha)}\{0\} \times \prod_{x\in[\alpha,1)}[0,1], & \alpha \in (0,1). \end{cases} \tag{A.3}$$

Thus $P(u) = P_0(u) = (0,1)$.

For $x \in (0,1]$, $n = 1, 2, \ldots$, define $u_{x,n} \in F^1_{USC}([0,3])$ as follows

$$u_{x,n}(t) = \begin{cases} 1, & t = 0, \\ \dfrac{-x}{n}t + x, & t \in (0,1], \\ 0, & t \notin [0,1]. \end{cases}$$

We can see that for each $x \in (0,1]$, $H_{\text{send}}(u_{x,n}, u_x) \to 0$ as $n \to \infty$.

Put

$$u_n = \prod_{x\in(0,1]} u_{x,n}.$$

We affirm that

(a-i) $[u]_0 = [u_n]_0 = \{\xi = (\xi_x)_{x\in(0,1]} \in \prod_{x\in(0,1]}[0,1] : \xi_x \to 0 \text{ as } x \to 0\}$ for $n = 1, 2, \ldots$;

(a-ii) $H_{\text{send}}(u_n, u) \to 0$;

(a-iii) For each $\alpha \in (0,1]$, there is a $\zeta \in [u]_\alpha$ such that $d(\zeta, [u_n]_\alpha) = 1$ for all $n = 1, 2, \ldots$;

(a-iv) $\{[u_n]_\alpha\}$ does not Kuratowski converge to $[u]_\alpha$ according to $(\prod_{x\in(0,1]}[0,3], d)$ when $\alpha \in (0,1]$;

(a-v) $H([u_n]_\alpha, [u]_\alpha) = 1$ for all $\alpha \in (0,1]$ and $n = 1, 2, \ldots$.

Set $D := \{\xi = (\xi_x)_{x\in(0,1]} \in \prod_{x\in(0,1]}[0,1] : \xi_x \to 0 \text{ as } x \to 0\}$. Since $[u_n]_\alpha \subseteq [u_{n+1}]_\alpha \subseteq [u]_\alpha$ for $\alpha \in (0,1]$ and $n = 1, 2, \ldots$, then we have that $[u_n]_0 \subseteq [u_{n+1}]_0 \subseteq [u]_0$ for $n = 1, 2, \ldots$. So to show (a-i), we only need to show that $D \subseteq [u_1]_0$ and $[u]_0 \subseteq D$.

To show $D \subseteq [u_1]_0$, it suffices to show that for each $\xi \in D$ and $\varepsilon > 0$, there exists an $\eta \in \cup_{\alpha>0}[u_1]_\alpha$ such that $d(\xi, \eta) < \varepsilon$.

Let $\xi \in D$ and $\varepsilon > 0$, then there is a $\delta > 0$ such that $\xi_x \le \varepsilon$ when $x < \delta$. Take an $\gamma \in (0,1]$ such that $[u_{1,\delta}]_\gamma \supseteq [0, 1-\varepsilon]$. Then $[u_{1,x}]_\gamma \supseteq [0, 1-\varepsilon]$ when $x \ge \delta$. So for each $x \in [\delta, 1]$, there is an $\eta(x) \in [u_{1,x}]_\gamma$ such that $|\xi_x - \eta(x)| \le \varepsilon$.

Define $\eta = (\eta_x)_{x\in(0,1]}$ as

$$\eta_x = \begin{cases} \eta(x), & x \ge \delta, \\ 0, & x < \delta. \end{cases}$$

Then for each $x \in (0,1]$, $\eta_x \in [u_{1,x}]_\gamma$ and therefore $\eta \in [u_1]_\gamma \subset \cup_{\alpha>0}[u_1]_\alpha$ And $d(\xi, \eta) = \sup_{x\in(0,1]} d(\xi_x, \eta_x) \le \varepsilon$.

To show $[u]_0 \subseteq D$, it suffices to show that if $\xi \notin D$, then $\xi \notin [u]_0$. Since $[u]_0 \subseteq \prod_{x\in(0,1]}[0,1]$, to verify this, it is enough to show that if $\xi \in \prod_{x\in(0,1]}[0,1] \setminus D$, then $\xi \notin [u]_0$.

Let $\xi \in \prod_{x\in(0,1]}[0,1] \setminus D$. Then there is an $\varepsilon > 0$ and a sequence $\{x_n\}$ in $[0,1]$ such that $x_n \to 0$ and $\xi_{x_n} \geq \varepsilon$. We claim that for each $\alpha > 0$ and $\zeta \in [u]_\alpha$, $d(\xi,\zeta) \geq \varepsilon$ because $\zeta_x = 0$ for $x < \alpha$. Thus $\xi \notin [u]_0 = \overline{\cup_{\alpha>0}[u]_\alpha}$.

To show (a-ii), we only need to show that

$$H^*(\mathrm{send}\, u_n, \mathrm{send}\, u) \to 0, \tag{A.4}$$

$$H^*(\mathrm{send}\, u, \mathrm{send}\, u_n) \to 0. \tag{A.5}$$

For $n = 1, 2, \ldots$, since $\mathrm{send}\, u_n \subset \mathrm{send}\, u$, then

$$H^*(\mathrm{send}\, u_n, \mathrm{send}\, u) = 0.$$

Thus (A.4) is true.

To show (A.5), let $(\xi, \alpha) \in \mathrm{send}\, u$, where $\xi = (\xi_x)_{x\in(0,1]}$. Put $\alpha_n = \max\{\alpha - 1/n, 0\}$. We claim that $(\xi, \alpha_n) \in \mathrm{send}\, u_n$.

If $\alpha_n = 0$, then by affirmation (a-i), $(\xi, \alpha_n) \in \mathrm{send}\, u_n$.

If $\alpha_n > 0$. Note that for each $n = 1, 2, \ldots$, $x \in (0,1]$ and $t \in [0,1]$, $u_{x,n}(t) \geq u_x(t) - \frac{1}{n}$. Then by Theorem Appendix A.3

$$u_n(\xi) = \inf_{x\in(0,1]} u_{x,n}(\xi_x) \geq \inf_{x\in(0,1]} u_x(\xi_x) - \frac{1}{n} = u(\xi) - \frac{1}{n} \geq \alpha - \frac{1}{n} = \alpha_n.$$

Thus $(\xi, \alpha_n) \in \mathrm{send}\, u_n$.

Since $|\alpha - \alpha_n| \leq 1/n$, then $d((\xi,\alpha), \mathrm{send}\, u_n) \leq d((\xi,\alpha),(\xi,\alpha_n)) \leq 1/n$. From the arbitrariness of $(\xi,\alpha) \in \mathrm{send}\, u$, we have $H^*(\mathrm{send}\, u, \mathrm{send}\, u_n) \leq 1/n$ (in fact, $H^*(\mathrm{send}\, u, \mathrm{send}\, u_n) = 1/n$) and thus (A.5) is true.

To show (a-iii), let $\alpha \in (0,1]$. Define $\zeta = (\zeta_x)_{x\in(0,1]}$ by

$$\zeta_x = \begin{cases} 1, & x = \alpha, \\ 0, & x = (0,1] \setminus \{\alpha\}. \end{cases}$$

Then by (A.3), $\zeta \in [u]_\alpha = \prod_{x\in(0,1]}[u_x]_\alpha$.

Note that for each $n = 1, 2, \ldots$ and $x \in (0,1]$, it holds that $[u_{x,n}]_x = \{0\}$. Then for each $n = 1, 2, \ldots$ and $\xi \in [u_n]_\alpha = \prod_{x\in(0,1]}[u_{x,n}]_\alpha$, we have $\xi_\alpha = 0$. and thus for $n = 1, 2, \ldots$,

$$d(\zeta, [u_n]_\alpha) = 1.$$

Hence (a-iii) is true.

(a-iv) follows immediately from (a-iii).

(a-v) follows immediately from (a-iii) and the fact that $[u]_\alpha \subset \prod_{x\in(0,1]}[0,1]$ and $[u_n]_\alpha \subset \prod_{x\in(0,1]}[0,1]$, $n = 1, 2, \ldots$.

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